

UNIFORMLY QUASICONFORMAL PARTIALLY HYPERBOLIC SYSTEMS

CLARK BUTLER AND DISHENG XU

ABSTRACT. We study smooth volume-preserving perturbations of the time-1 map of the geodesic flow ψ_t of a closed Riemannian manifold of dimension at least three with constant negative curvature. We show that such a perturbation has equal extremal Lyapunov exponents with respect to volume within both the stable and unstable bundles if and only if it embeds as the time-1 map of a smooth volume-preserving flow that is smoothly orbit equivalent to ψ_t . Our techniques apply more generally to give an essentially complete classification of smooth, volume-preserving, dynamically coherent partially hyperbolic diffeomorphisms which satisfy a uniform quasiconformality condition on their stable and unstable bundles and have either uniformly compact center foliation or are obtained as perturbations of the time-1 map of an Anosov flow.

1. INTRODUCTION

A surprising number of rigidity problems originally posed in negatively curved geometry turn out to have solutions that are dynamical in nature. We review one such story here: Sullivan proposed, following work of Gromov[17] and Tukia[33], that closed Riemannian manifolds of constant negative curvature and dimension at least 3 should be characterized up to isometry by the property that the geodesic flow acts *uniformly quasiconformally* on the unstable foliation[32]. Informally, the uniform quasiconformality property states that the flow does not distort the shape of metric balls inside of a given horosphere over a long period of time. Sullivan's conjecture was partially confirmed by the work of Kanai [23] who showed that among contact Anosov flows the geodesic flows of constant negative curvature manifolds are characterized up to C^1 orbit equivalence by a uniform quasiconformality. Later the minimal entropy rigidity theorem of Besson, Courtois, and Gallot [5] completed the proof of Sullivan's conjecture among many other outstanding conjectures in negatively curved geometry.

From a geometric perspective this completes the story, but from a dynamical perspective this raises many new questions. Already in the work of Kanai we see that the dynamical version of this rigidity result holds for a larger class of Anosov flows than just geodesic flows. Sadovskaya initiated a program to extend these results further to smooth volume-preserving Anosov flows and diffeomorphisms [29], which was completed in a series of works by Fang ([12], [13], [14]) who obtained the following remarkable result: all smooth volume-preserving Anosov flows which are uniformly quasiconformal on the stable and unstable foliation are smoothly orbit equivalent either to the suspension of a hyperbolic toral automorphism or the geodesic flow on the unit tangent bundle of a constant negative curvature closed Riemannian manifold. Thus we see that not even the contact structure of the flow is necessary to obtain dynamical rigidity for uniformly quasiconformal Anosov flows.

In a different direction one can ask whether the uniform quasiconformality condition can be relaxed to a condition that is more natural from the perspective of ergodic theory. This direction was pursued by the first author, who showed that for geodesic flows of $\frac{1}{4}$ -pinched negatively curved manifolds, uniform quasiconformality can be derived from the significantly weaker dynamical condition of equality of all Lyapunov exponents with respect to volume on the unstable bundle [10].

Our principal goal is to show that for all of the rigidity phenomena derived from uniform quasiconformality above, *not even the structure of an Anosov flow is necessary*. Let us be more precise: consider a closed Riemannian manifold X of constant negative curvature with $\dim X \geq 3$. Let T^1X be the unit tangent bundle of X and let $\psi_t : T^1X \rightarrow T^1X$ denote the time- t map of the geodesic flow. This flow preserves a smooth volume m on T^1X known as the Liouville measure. Consider *any* smooth diffeomorphism f which is C^1 -close to the time-1 map ψ_1 and which preserves the volume m . By the work of Hirsch, Pugh, and Shub [19], f is *partially hyperbolic*, meaning that there is a Df -invariant splitting $T(T^1X) = E^u \oplus E^c \oplus E^s$ where E^u is exponentially expanded by Df , E^s is exponentially contracted by Df , and the behavior of Df on the 1-dimensional center direction E^c (which is close to the flow direction for ψ_t) is dominated by the expansion and contraction on E^u and E^s respectively. We give a more precise definition in Section 2. We then choose a continuous norm $\|\cdot\|$ on E^u and define the *extremal Lyapunov exponents* of f on E^u by

$$\lambda_+^u(f) = \inf_{n \geq 1} \int_M \log \|Df^n|E^u\| dm,$$

$$\lambda_-^u(f) = \sup_{n \geq 1} \int_M \log \|(Df^n|E^u)^{-1}\|^{-1} dm.$$

We define $\lambda_+^s(f)$ and $\lambda_-^s(f)$ similarly with E^s replacing E^u .

THEOREM 1. *There is a C^2 -open neighborhood \mathcal{U} of ψ_1 in the space of C^∞ volume-preserving diffeomorphisms of T^1X such that if $f \in \mathcal{U}$ and both of the equalities $\lambda_+^u(f) = \lambda_-^u(f)$ and $\lambda_+^s(f) = \lambda_-^s(f)$ hold then there is a C^∞ volume-preserving flow φ_t with $\varphi_1 = f$. Furthermore φ_t is smoothly orbit equivalent to ψ_t .*

This theorem improves on the techniques used in the previous rigidity theorems in several fundamental ways. We are able to deduce uniform quasiconformality of the action of Df on E^u and E^s from equality of the extremal Lyapunov exponents entirely outside of the geometric context considered in [10] by using new methods. We then use this uniform quasiconformality to completely reconstruct the smooth flow φ_t in which f embeds as the time-1 map. We emphasize that for a typical perturbation f of ψ_1 the foliation \mathcal{W}^c tangent to E^c (which is our candidate for the flowlines of φ_t) is only a continuous foliation of T^1X with no transverse smoothness properties. This is one of the many reasons that strong rigidity results in the realm of partially hyperbolic diffeomorphisms are quite rare. Our inspiration was an impressive rigidity theorem of Avila, Viana and Wilkinson which overcame this obstacle to show that if we take X to be a negatively curved surface instead and f a C^1 -small enough C^∞ volume-preserving perturbation of the time-1 map ψ_1 such that the center foliation of f is absolutely continuous, then f is also the time-1 map of a smooth volume-preserving flow [3]. Our result can be viewed in an appropriate sense as the higher dimensional analogue of this theorem.

We now explain the organization of the paper. The techniques used in the proof of Theorem 1 have much more general applications which can also be applied to the study of C^∞ volume-preserving partially hyperbolic diffeomorphisms which satisfy a uniform quasiconformality condition on their stable and unstable bundles and either have uniformly compact center foliation or are obtained as a perturbation of the time-1 map of an Anosov flow. These results are stated in Theorems 2 and 4 and Corollary 3 of Section 2 after we introduce some necessary terminology. In Section 3 we show that under a Lyapunov stability type assumption on the action of a partially hyperbolic diffeomorphism f on its center foliation, uniform quasiconformality implies that the holonomy maps of the center-stable and center-unstable foliations of f are quasiconformal. We use this to show that the center foliation of f is absolutely continuous. In Section 4 we prove Theorems 2 and 4 and Corollary 3 by using the quasiconformality obtained in Section 3 to improve the regularity of the center-stable and center-unstable holonomy maps to show that these maps are actually C^∞ . In Section 5 we finish the proof of Theorem 1 by deducing uniform quasiconformality from the condition of equality of extremal Lyapunov exponents. The arguments in Section 5 do not rely on the results of Sections 3 and 4 and may be read independently of the rest of the paper.

Acknowledgments: We thank Amie Wilkinson for numerous useful discussions regarding the content of this paper. These discussions resulted in significant simplification of the proof of Theorem 1. The second author would like to thank his director of thesis Professor Artur Avila for his supervision and encouragement. This work was partially completed while both authors were visiting the Instituto Nacional de Matemática Pura e Aplicada, the second author being supported by *réseau franco-brésilien en mathématiques*. The first author was supported by the National Science Foundation Graduate Research Fellowship under Grant # DGE-1144082.

2. STATEMENT OF RESULTS

A C^1 diffeomorphism $f : M \rightarrow M$ of a closed Riemannian manifold M is *partially hyperbolic* if there is a Df -invariant splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle of M such that for some $k \geq 1$, any $x \in M$, and any choice of unit vectors $v^s \in E_x^s$, $v^c \in E_x^c$, $v^u \in E_x^u$,

$$\begin{aligned} \|Df^k(v^s)\| &< 1 < \|Df^k(v^u)\|, \\ \|Df^k(v^s)\| &< \|Df^k(v^c)\| < \|Df^k(v^u)\|. \end{aligned}$$

By modifying the Riemannian metric on M if necessary we can always assume $k = 1$ in the above definition. We will always require that the bundles E^s and E^u are nontrivial. We will also always require that M is connected. We define for $x \in M$, $n \in \mathbb{Z}$,

$$K^u(x, n) = \frac{\sup\{\|Df^n(v^u)\| : v^u \in E^u(x), \|v^u\| = 1\}}{\inf\{\|Df^n(v^u)\| : v^u \in E^u(x), \|v^u\| = 1\}},$$

and define $K^s(x, n)$ similarly with E^u replaced by E^s . The quantities K^u and K^s measure the failure of the iterates of Df to be conformal on the bundles E^u and E^s respectively. We say that f is *uniformly u -quasiconformal* if $\dim E^u \geq 2$ and K^u is uniformly bounded in x and n . Similarly we say that f is *uniformly s -quasiconformal* if $\dim E^s \geq 2$ and K^s is uniformly bounded in x and n . If f is both uniformly

u -quasiconformal and s -quasiconformal then we simply say that f is *uniformly quasiconformal*.

Our definition of uniform quasiconformality for partially hyperbolic systems extends previous definitions of uniform quasiconformality which were considered for Anosov diffeomorphisms and Anosov flows. If the center bundle E^c is trivial or if f embeds as the time-1 map of an Anosov flow (so that E^c is tangent to the flow direction) then these definitions reduce to the standard notions of uniform quasiconformality for Anosov systems defined by Sadovskaya [29]. If $\dim E^u = 1$ then $K^u \equiv 1$ for any choice partially hyperbolic f , so the boundedness of K^u does not give new information about f . This is the reason we require $\dim E^u \geq 2$ in the definition of uniform u -quasiconformality; the uniform quasiconformality conditions are only interesting when the bundles in question have dimension at least 2.

We define a C^∞ diffeomorphism f to be *volume-preserving* there is an f -invariant probability measure m on M which is smoothly equivalent to the Riemannian volume. It is not hard to show using Kingman's subadditive ergodic theorem [25] that when f is ergodic with respect to m we have

$$\lim_{n \rightarrow \infty} \frac{\log K^u(x, n)}{n} = \lambda_+^u(f) - \lambda_-^u(f) \text{ for } m\text{-a.e. } x \in M.$$

We refer to [22] for more details on this equality. Thus asymptotic subexponential growth of K^u is equivalent to the equality $\lambda_+^u(f) = \lambda_-^u(f)$. Theorem 1 asks in part for the deduction of a uniform bound $K^u(x, n) \leq C$ from this asymptotic subexponential growth condition.

Fang proved that all volume-preserving C^∞ uniformly quasiconformal diffeomorphisms are C^∞ conjugate to a hyperbolic toral automorphism [12]. This generalized the classification result of Sadovskaya which held under the additional assumption that f was symplectic [29]. Theorem 2 and Corollary 3 below extend this classification to cover a certain C^1 -open set of C^∞ volume-preserving partially hyperbolic diffeomorphisms. Before stating these theorems we need to introduce a few more basic notions from partially hyperbolic dynamics.

We refer the reader to [9] for a deeper discussion of partial hyperbolicity and the properties that follow. We will assume for the rest of the paper that f is C^∞ . Then the bundles E^s and E^u are tangent to foliations \mathcal{W}^s and \mathcal{W}^u known respectively as the stable and unstable foliations. These foliations have C^∞ leaves but the distributions E^s and E^u which they are tangent to are themselves typically only Hölder continuous. We say that f is *dynamically coherent* if there are also f -invariant foliations \mathcal{W}^{cs} and \mathcal{W}^{cu} with C^1 leaves which are tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively. It follows by intersecting these two foliations that there is an f -invariant foliation \mathcal{W}^c with C^1 leaves which is tangent to E^c .

For $r \geq 1$ we write that a map is $C^{r+\alpha}$ if it is C^r and the r th-order derivatives are uniformly Hölder continuous of exponent $\alpha > 0$. For a foliation \mathcal{W} of an n -dimensional smooth manifold M by k -dimensional submanifolds we define \mathcal{W} to be a $C^{r+\alpha}$ foliation if for each $x \in M$ there is an open neighborhood V_x of x and a $C^{r+\alpha}$ diffeomorphism $\Psi_x : V_x \rightarrow D^k \times D^{n-k} \subset \mathbb{R}^n$ (where D^j denotes the ball of radius 1 centered at 0 in \mathbb{R}^j) such that Ψ_x maps \mathcal{W} to the standard smooth foliation of $D^k \times D^{n-k}$ by k -disks $D^k \times \{y\}$, $y \in D^{n-k}$. This is the notion of regularity of a

foliation considered by Pugh, Shub, and Wilkinson in their analysis of regularity properties of invariant foliations for partially hyperbolic systems [27].

We say that f is r -bunched if there is some $k \geq 1$ such that for any unit vectors $v^s \in E_x^s$, $v^c \in E_x^c$, $v^u \in E_x^u$,

$$\|Df^k(v^s)\| \|Df^k(v^c)\|^r < 1, \quad \|Df^k(v^u)\| \|Df^k(v^c)\|^r < 1,$$

and furthermore for any pair of unit vectors $v_1^c, v_2^c \in E^c(x)$,

$$\frac{\|Df^k(v^s)\| \cdot \|Df^k(v_1^c)\|^r}{\|Df^k(v_2^c)\|} < 1, \quad \frac{\|Df^k(v^u)\| \cdot \|Df^k(v_1^c)\|^r}{\|Df^k(v_2^c)\|} < 1.$$

The case $r = 1$ corresponds to the *center-bunching* condition considered by Burns and Wilkinson in their proof of the ergodicity of accessible, volume-preserving, center-bunched C^2 partially hyperbolic diffeomorphisms [9]. When f is smooth and dynamically coherent, the r -bunching inequalities imply that the foliations \mathcal{W}^{cs} and \mathcal{W}^{cu} have uniformly $C^{r+\alpha}$ leaves for some $\alpha > 0$ [27]. We say that f is ∞ -bunched if it is r -bunched for every $r \geq 1$. If f is ∞ -bunched and dynamically coherent then the leaves of \mathcal{W}^{cs} and \mathcal{W}^{cu} are C^∞ . A natural situation in which the ∞ -bunching condition holds is when there is a continuous Riemannian metric on E^c with respect to which $Df|_{E^c}$ is an isometry. More generally if f is center bunched, accessible, and volume-preserving and all of the Lyapunov exponents of f with respect to volume on E^c are zero, then by the results of Kalinin and Sadovskaya f is ∞ -bunched [22].

Finally, when f is dynamically coherent we say that the center foliation \mathcal{W}^c is *uniformly compact* if all of the leaves of \mathcal{W}^c are compact with a uniform bound on their diameters in the induced Riemannian metric on E^c from TM .

THEOREM 2. *Let f be a C^∞ dynamically coherent volume-preserving partially hyperbolic diffeomorphism. Suppose that f is uniformly quasiconformal, r -bunched for some $r \geq 1$, and has uniformly compact center foliation. Then*

- (1) *There is an $\alpha > 0$ such that \mathcal{W}^{cs} , \mathcal{W}^c , and \mathcal{W}^{cu} are $C^{r+\alpha}$ foliations of M and both \mathcal{W}^s and \mathcal{W}^u are $C^{1+\alpha}$ foliations of M ,*
- (2) *There is a closed C^r Riemannian manifold N , a $C^{r+\alpha}$ submersion $\pi : M \rightarrow N$ with fibers given by the \mathcal{W}^c foliation, and a $C^{r+\alpha}$ volume-preserving uniformly quasiconformal Anosov diffeomorphism $g : N \rightarrow N$ such that $g \circ \pi = \pi \circ f$.*
- (3) *If f is ∞ -bunched then the statements of (1) and (2) are true with $r = \infty$. Furthermore \mathcal{W}^s and \mathcal{W}^u are also C^∞ foliations of M and g may be taken to be a hyperbolic automorphism of a torus N .*

When the center bundle of f is one-dimensional we can derive a sharper result as a corollary. We define a smooth diffeomorphism $f : M \rightarrow M$ to be an *isometric extension* of another smooth diffeomorphism $g : N \rightarrow N$ if there is a smooth submersion $\pi : M \rightarrow N$ satisfying $g \circ \pi = \pi \circ f$ and such that this submersion has compact fibers and there is a smoothly varying family of Riemannian metrics $\{d_x\}_{x \in N}$ on the fibers $\{\pi^{-1}(x)\}_{x \in N}$ such that the induced maps $f_x : \pi^{-1}(x) \rightarrow \pi^{-1}(g(x))$ are isometries with respect to these metrics.

COROLLARY 3. *Let f be a C^∞ dynamically coherent volume-preserving partially hyperbolic diffeomorphism with $\dim E^c = 1$. Suppose that f is uniformly quasiconformal and that f has uniformly compact center foliation. Then f is an isometric extension of a hyperbolic toral automorphism.*

We make some comments on Theorem 2 and Corollary 3 before proceeding. If $M = N \times S$ for pair of compact smooth manifolds N and S , $g_0 : N \rightarrow N$ is an Anosov diffeomorphism and $f_0 : M \rightarrow M$ is a smooth extension of g_0 such that f_0 is an r -bunched volume-preserving partially hyperbolic diffeomorphism ($r \geq 1$) with center leaves of the form $\mathcal{W}^c(x, s) = \{x\} \times S$ for $(x, s) \in N \times S$, then the center leaves of f_0 are normally hyperbolic and uniformly compact. Thus there is a C^1 open neighborhood \mathcal{U} of f_0 in the space of C^∞ volume-preserving diffeomorphisms of M such that if $f \in \mathcal{U}$ then f is dynamically coherent and has uniformly compact center foliation. This follows from the theory of normally hyperbolic invariant manifolds developed by Hirsch, Pugh, and Shub [19]. Hence, with the exception of the uniform quasiconformality hypothesis, the hypotheses of Theorem 2 and Corollary 3 are not particularly restrictive among partially hyperbolic diffeomorphisms.

The limiting factor for the smoothness of the foliations \mathcal{W}^{cs} and \mathcal{W}^{cu} in Theorem 2 turns out to be the regularity of the leaves of the foliations themselves. Corollary 26 below shows that the holonomy maps of \mathcal{W}^{cs} and \mathcal{W}^{cu} between local unstable/local stable leaves respectively are C^∞ . In fact they are analytic maps in an appropriate choice of coordinates. The r -bunching inequalities in the hypotheses of Theorem 2 are only required to obtain that the leaves of the foliations \mathcal{W}^{cs} and \mathcal{W}^{cu} are $C^{r+\alpha}$; they are never used directly in the proof.

The classification results of Sadovskaya and Fang for C^∞ volume-preserving uniformly quasiconformal Anosov diffeomorphisms rely fundamentally on the smooth conjugacy classification theorem of Benoist and Labourie for Anosov diffeomorphisms with C^∞ stable and unstable foliations [4]. However, the regularity of the uniformly quasiconformal Anosov diffeomorphism g obtained from Theorem 2 is limited by the regularity of the center foliation, which in turn is limited by the r -bunching hypothesis. The most we can obtain with our methods is that g is $C^{r+\alpha}$. This is the reason we can only derive the stronger results of part (3) of Theorem 2 under the ∞ -bunching hypothesis on f .

Finally we observe that the conclusions of Theorem 2 imply in particular that the center foliation of f is absolutely continuous with respect to volume. We refer to Definition 12 below for our definition of absolute continuity of foliation. Pugh and Wilkinson showed that an isometric extension of a hyperbolic automorphism of the two-dimensional torus \mathbb{T}^2 can be perturbed to make the center Lyapunov exponent nonzero and thus cause the center foliation to fail to be absolutely continuous [30]. Corollary 3 shows that it is not possible to make such a perturbation of an isometric extension of a uniformly quasiconformal hyperbolic automorphism of a higher dimensional torus which maintains uniform quasiconformality on both the stable and unstable bundles.

For our next theorem we consider partially hyperbolic diffeomorphisms which are obtained as perturbations of the time-1 maps of Anosov flows. Let $\psi_t : M \rightarrow M$ be a C^∞ volume-preserving Anosov flow with stable and unstable bundles of dimension at least 2.

THEOREM 4. *Suppose that there is a finite cover \hat{M} of M such that the lift of ψ_t to an Anosov flow $\hat{\psi}_t : \hat{M} \rightarrow \hat{M}$ has no periodic orbits of period ≤ 2 .*

Then there is a C^1 -open neighborhood \mathcal{U} of ψ_1 in the space of volume-preserving C^∞ diffeomorphisms of M such that if $f \in \mathcal{U}$ and f is uniformly quasiconformal then the

invariant foliations \mathcal{W}^{cs} , \mathcal{W}^c , and \mathcal{W}^{cu} of f are C^∞ and there is a C^∞ volume-preserving uniformly quasiconformal Anosov flow $\varphi_t : M \rightarrow M$ with $\varphi_1 = f$.

Before making further comments on this theorem we recall the notion of orbit equivalence of Anosov flows. Two C^∞ Anosov flows $\varphi_t, \psi_t : M \rightarrow M$ are C^r orbit equivalent ($r \in [0, \infty]$) if there is a C^r map $h : M \rightarrow M$ such that for every $x \in M$ and $t \in \mathbb{R}$, $h(\varphi_t(x))$ lies on the ψ_t -orbit of $h(x)$.

From the classification of C^∞ volume-preserving uniformly quasiconformal Anosov flows obtained by Fang [14] we conclude that the flow φ_t obtained in the conclusion of Theorem 4 is C^∞ orbit equivalent either to the suspension flow of a hyperbolic toral automorphism or the geodesic flow on the unit tangent bundle of a constant negative curvature Riemannian manifold.

The hypothesis in Theorem 4 that there is a finite cover \hat{M} for which the lift $\hat{\psi}_t$ has no periodic orbits of period ≤ 2 is very mild. It always holds if ψ_t is C^0 orbit equivalent to the suspension flow of an algebraic Anosov diffeomorphism or the geodesic flow of a closed negatively curved Riemannian manifold. We expect that Theorem 4 holds without this hypothesis, however this hypothesis does simplify some constructions in the proofs, particularly in Section 4.1.

We recall now the definitions of su -paths and accessibility for a partially hyperbolic diffeomorphism which will play a crucial role in the proof of Theorem 1 in Section 5. For a partially hyperbolic diffeomorphism $f : M \rightarrow M$ an su -path in M is a piecewise C^1 curve γ in M such that γ decomposes into finitely many C^1 sub-curves $\gamma_{x_i x_{i+1}}$ connecting x_i to x_{i+1} and such that each curve $\gamma_{x_i x_{i+1}}$ is contained in a single \mathcal{W}^s or \mathcal{W}^u leaf. We define f to be *accessible* if any two points in M can be joined by an su -path.

A notable aspect of Theorems 2 and 4 and Corollary 3 is that their hypotheses do not include any accessibility or ergodicity assumptions on f with respect to the volume m . This requires us to take some additional care at certain points in the proof. The accessibility hypotheses is used strongly in the rigidity theorem of Avila-Viana-Wilkinson and ergodicity with respect to volume is used in the classification results of Sadovskaya and Fang.

The results of Corollary 3 and Theorem 4 suggest that it may be possible to obtain a global smooth classification of C^∞ volume-preserving, dynamically coherent, uniformly quasiconformal partially hyperbolic diffeomorphisms with one-dimensional center in terms of the classification of uniformly quasiconformal Anosov diffeomorphisms and Anosov flows. We give an example which illustrates some of the difficulties in obtaining a classification beyond these theorems.

Consider the 5×5 integer matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & -3 & 1 \end{pmatrix},$$

and let $f_A : \mathbb{T}^5 \rightarrow \mathbb{T}^5$ be the induced linear map of A on the 5-torus $\mathbb{T}^5 = \mathbb{R}^5 / \mathbb{Z}^5$. By numerical computation the five complex eigenvalues of A satisfy

$$|\lambda_1| = |\lambda_2| > |\lambda_3| > 1 > |\lambda_4| = |\lambda_5|,$$

$$\overline{\lambda_1} = \lambda_2 \notin \mathbb{R},$$

$$\overline{\lambda_3} = \lambda_4 \notin \mathbb{R}.$$

Thus f_A is a hyperbolic toral automorphism which may also be viewed as a partially hyperbolic diffeomorphism with splitting $T\mathbb{T}^5 = E^u \oplus E^c \oplus E^s$, where E^u is the real part of the complex eigenspaces corresponding to the pair of conjugate complex eigenvalues λ_1 and λ_2 , E^c is the eigenspace corresponding to λ_3 , and E^s is the real part of the complex eigenspaces corresponding to λ_4 and λ_5 . We conclude f_A is a smooth, volume-preserving, dynamically coherent uniformly quasiconformal partially hyperbolic diffeomorphism with one-dimensional center.

We then pose the following problem,

PROBLEM 5. *Is there a C^1 -open neighborhood \mathcal{U} of f_A in the space of smooth volume-preserving diffeomorphisms of \mathbb{T}^5 such that if $f \in \mathcal{U}$ is uniformly quasiconformal then the invariant foliations \mathcal{W}^{cs} , \mathcal{W}^c , and \mathcal{W}^{cu} of f are smooth?*

We expect the answer to Problem 5 to be “no” but the difficulty of constructing nontrivial uniformly quasiconformal perturbations of f_A is a significant obstruction to confirming our suspicions. We note that each $f \in \mathcal{U}$ is an Anosov diffeomorphism if \mathcal{U} is chosen small enough.

3. QUASICONFORMALITY OF THE CENTER HOLONOMY

We fix M to be a closed Riemannian manifold with distance d and let $f : M \rightarrow M$ be a C^∞ dynamically coherent partially hyperbolic diffeomorphism. For $* \in \{s, c, u, cu, cs\}$ we let d_* denote the induced Riemannian metric on the leaves of the foliation \mathcal{W}^* . We write $\mathcal{W}^*(x)$ for the leaf of \mathcal{W}^* passing through $x \in M$. We write diam_* for the diameter of a subset of \mathcal{W}^* measured with respect to the d_* metric. For $r > 0$ we write $\mathcal{W}_r^*(x)$ for the open ball of radius r in $\mathcal{W}^*(x)$ centered at x in the d_* metric.

We can find small constants $R \geq r > 0$ with the property that for any $x \in M$, $y \in \mathcal{W}_r^{cs}(x)$ and $z \in \mathcal{W}_r^u(x)$ the local leaves $\mathcal{W}_r^{cs}(z)$ and $\mathcal{W}_R^u(y)$ intersect in exactly one point which we denote by $h_{xy}^{cs}(z)$. This defines the *local center-stable holonomy* map between local unstable leaves of f . Similarly we require that if $x \in M$, $y \in \mathcal{W}_r^{cu}(x)$ and $z \in \mathcal{W}_r^s(x)$ then the local leaves $\mathcal{W}_r^{cu}(z)$ and $\mathcal{W}_R^s(y)$ intersect in exactly one point which we denote by $h_{xy}^{cu}(z)$, and use this to define the *local center-unstable holonomy*.

We introduce some useful shorthand related to these holonomy maps. The center-stable holonomy maps and center-unstable holonomy maps will sometimes be referred to as *cs-holonomy* and *cu-holonomy* respectively. When the domain and range are understood we will omit the subscripts on h^{cs} and h^{cu} . We will write $\mathcal{W}_{loc}^*(x)$ for any open ball of the form $\mathcal{W}_t^*(x)$ with $r \leq t \leq R$. Hence it makes sense in our shorthand to write $h^{cs} : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y)$ for the *cs-holonomy* maps.

Our starting point is the following non-stationary smooth linearization lemma of Sadovskaya applied to the unstable foliation \mathcal{W}^u which is uniformly contracted by f^{-1} ,

PROPOSITION 6. [29, Proposition 4.1] *Suppose that f is a C^∞ uniformly u -quasiconformal partially hyperbolic diffeomorphism. Then for each $x \in M$ there is a C^∞ diffeomorphism $\Phi_x : E_x^u \rightarrow \mathcal{W}^u(x)$ satisfying*

- (1) $\Phi_x \circ Df_x = f \circ \Phi_{f(x)}$,
- (2) $\Phi_x(0) = x$ and $D_0\Phi_x$ is the identity map,

- (3) The family of diffeomorphisms $\{\Phi_x\}_{x \in M}$ varies continuously with x in the C^∞ topology.

The family $\{\Phi_x\}_{x \in M}$ satisfying (1), (2), and (3) is unique.

The bundle E^u is a Hölder continuous subbundle of TM with some Hölder exponent $\beta > 0$ [27]. Therefore the restriction $Df|_{E^u}$ of the derivative of f to the unstable bundle is a Hölder continuous linear cocycle over f in the sense of Kalinin-Sadovskaya [22]. For $x, y \in M$ two nearby points we let $I_{xy} : E_x^u \rightarrow E_y^u$ be a linear identification which is β -Hölder close to the identity. The diffeomorphism f is uniformly u -quasiconformal if and only if, in the terminology of [22], the cocycle $Df|_{E^u}$ is uniformly quasiconformal. The following proposition thus applies to $Df|_{E^u}$,

PROPOSITION 7. [22, Proposition 4.2] For $y \in \mathcal{W}_{loc}^u(x)$, the limit

$$\lim_{n \rightarrow \infty} Df_{f^{-n}y}^n \circ I_{f^{-n}x f^{-n}y} \circ Df_x^{-n}|_{E^u} := H_{xy}^u,$$

exists uniformly in x and y and defines a linear map from E_x^u to E_y^u with the following properties for $x, y, z \in M$,

- (1) $H_{xx}^u = \text{Id}$ and $H_{yz}^u \circ H_{xy}^u = H_{xz}^u$;
- (2) $H_{xy}^u = Df_{f^{-n}y}^n \circ H_{f^{-n}x f^{-n}y}^u \circ Df_x^{-n}$ for any $n \geq 0$.
- (3) $\|H_{xy}^u - I_{xy}\| \leq C d(x, y)^\beta$, β the exponent of Hölder continuity for E^u .

Furthermore H^u is the unique collection of linear identifications with these properties. Similarly if $y \in \mathcal{W}_{loc}^s(x)$ then the limit $\lim_{n \rightarrow \infty} Df_y^{-n} \circ I_{f^n x f^n y} \circ Df_x^n|_{E^u} := H_{xy}^s$ exists and gives a linear map from E_x^u to E_y^u with analogous properties. H^u and H^s are known as the unstable and stable holonomies of $Df|_{E^u}$ respectively.

Using property (2) of the unstable and stable holonomies of $Df|_{E^u}$ from Proposition 7 we may uniquely extend H^u and H^s to be defined for any $y \in \mathcal{W}^u(x)$ and any $y \in \mathcal{W}^s(x)$ respectively.

The transition maps between the charts given by Proposition 6 are affine with derivatives given by the unstable holonomy H^u ,

PROPOSITION 8. Suppose that f is uniformly u -quasiconformal and let $\{\Phi_x\}_{x \in M}$ be the charts of Proposition 6. Then for each $x \in M$ and $y \in \mathcal{W}^u(x)$ the map $\Phi_y^{-1} \circ \Phi_x : E_x^u \rightarrow E_y^u$ is an affine map with derivative H_{xy}^u .

Proof. For any $n \geq 0$ and any $v \in E_x^u$ we use the defining properties of the charts $\{\Phi_x\}_{x \in M}$ to write

$$\begin{aligned} D_v(\Phi_y^{-1} \circ \Phi_x) &= D_v(Df_{f^{-n}y}^n \circ \Phi_{f^{-n}y}^{-1} \circ \Phi_{f^{-n}x} \circ Df_x^{-n}) \\ &= Df_{f^{-n}y}^n \circ D_{Df^{-n}(v)}(\Phi_{f^{-n}y}^{-1} \circ \Phi_{f^{-n}x}) \circ Df_x^{-n}, \end{aligned}$$

We have a bound

$$\left\| D_{Df^{-n}(v)}(\Phi_{f^{-n}y}^{-1} \circ \Phi_{f^{-n}x}) - I_{f^{-n}x f^{-n}y} \right\| \leq C(v) d(f^{-n}(x), f^{-n}(y))^\beta,$$

with the constant $C(v)$ depending only on the distance of v from the origin in E_x^u , because the charts $\{\Phi_x\}_{x \in M}$ vary continuously in the C^∞ topology. From the existence of this bound and the proof of [22, Proposition 4.2] we conclude that

$$\lim_{n \rightarrow \infty} Df_{f^{-n}y}^n \circ D_{Df^{-n}(v)}(\Phi_{f^{-n}y}^{-1} \circ \Phi_{f^{-n}x}) \circ Df_x^{-n} = H_{xy}^u$$

This implies that $D_v(\Phi_y^{-1} \circ \Phi_x) = H_{xy}^u$ for every $v \in E_x^u$, from which it follows that $\Phi_y^{-1} \circ \Phi_x$ is an affine map from E_x^u to E_y^u with linear part H_{xy}^u . \square

We now set $k := \dim E^u$ and recall that our assumption that f is uniformly u -quasiconformal requires that $k \geq 2$. We recall the notion of a quasiconformal map between domains in \mathbb{R}^k where we equip \mathbb{R}^k with the Euclidean norm $\|\cdot\|$,

DEFINITION 9. Let $h : U \rightarrow V$ be a homeomorphism between two open subsets U, V of \mathbb{R}^k . The linear dilatation of h at $x \in U$ is defined to be

$$L_h(x) = \limsup_{r \rightarrow 0} \frac{\max_{\|y-x\|=r} \|f(y) - f(x)\|}{\min_{\|y-x\|=r} \|f(y) - f(x)\|}$$

For $K \geq 1$ we define h to be K -quasiconformal if $L_h(x) \leq K$ for every $x \in U$.

Each of the normed vector spaces E_x^u (with norm induced from the Riemannian metric on TM) carries the linear structure of \mathbb{R}^k with a norm that is uniformly comparable to the Euclidean norm on \mathbb{R}^k . Hence K -quasiconformality can also be defined for homeomorphisms between open subsets of E_x^u and E_y^u for $x, y \in M$. It is this sense of K -quasiconformality which is used in Lemma 10 below, which is the main result of this section.

LEMMA 10. Let f be a C^∞ dynamically coherent partially hyperbolic diffeomorphism. Suppose that f is uniformly u -quasiconformal and that there exists $C > 0$ such that if $y \in \mathcal{W}_{loc}^c(x)$ then

$$d_c(f^n(x), f^n(y)) \leq C, \text{ for every } n \geq 0.$$

Then there is a constant $K \geq 1$ such that for any two points $x \in M, y \in \mathcal{W}_{loc}^{cs}(x)$, the homeomorphism

$$\Phi_y^{-1} \circ h^{cs} \circ \Phi_x : \Phi_x^{-1}(\mathcal{W}_{loc}^u(x)) \rightarrow \Phi_y^{-1}(\mathcal{W}_{loc}^u(y)),$$

is K -quasiconformal.

Proof. By hypothesis there is a constant $C > 0$ such that for any $x \in M$ and $y \in \mathcal{W}_{loc}^c(x)$ we have $d_c(f^n(x), f^n(y)) \leq C$ for every $n \geq 0$. Since the stable foliation \mathcal{W}^s is contracted by f^n for $n \geq 0$, this implies that there is a (possibly larger) constant $C > 0$ such that for every $x \in M$ and $y \in \mathcal{W}_{loc}^{cs}(x)$,

$$d_{cs}(f^n(x), f^n(y)) \leq C, \text{ for every } n \geq 0.$$

The compactness of M and the continuity of the cs -holonomy maps implies uniform continuity of these holonomy maps between local leaves of \mathcal{W}^u which are a bounded distance apart. More precisely, for each $\zeta \geq 1$ there is a constant $L = L(\zeta) \geq 1$ such that for any $x, y, z \in M$ with $y \in \mathcal{W}^{cs}(x)$ and $z \in \mathcal{W}^u(x)$ such that $\zeta^{-1} \leq d_u(x, z) \leq \zeta$ and $d_{cs}(x, y) \leq C$ then

$$L(\zeta)^{-1} \leq d_u(y, h^{cs}(z)) \leq L(\zeta).$$

For each $n \geq 0$ we define the center-stable holonomy map $h_n^{cs} : f^n(\mathcal{W}_{loc}^u(x)) \rightarrow f^n(\mathcal{W}_{loc}^u(y))$ by $h_n^{cs} = f^n \circ h^{cs} \circ f^{-n}$, where h^{cs} is the local center-stable holonomy from $\mathcal{W}_{loc}^u(x)$ to $\mathcal{W}_{loc}^u(y)$. By the previous remark we conclude that if $z \in \mathcal{W}_{loc}^u(x)$ and $y \in \mathcal{W}_{loc}^{cs}(x)$ satisfies $\zeta^{-1} \leq d_u(f^n(z), f^n(x)) \leq \zeta$ for some $n \geq 0$ then

$$(3.1) \quad L(\zeta)^{-1} \leq d_u(f^n(y), h_n^{cs}(f^n(z))) \leq L(\zeta).$$

Now fix $x \in M$ and $y \in \mathcal{W}_{loc}^{cs}(x)$. We first estimate the dilatation of the homeomorphism $\Phi_y^{-1} \circ h^{cs} \circ \Phi_x$ at 0. Using the equivariance properties of the charts from Proposition 6, we have for every $n \geq 0$,

$$\begin{aligned}\Phi_y^{-1} \circ h^{cs} \circ \Phi_x &= \Phi_y^{-1} \circ f^{-n} \circ h_n^{cs} \circ f^n \circ \Phi_x \\ &= Df_y^{-n} \circ \Phi_{f^n(y)}^{-1} \circ h_n^{cs} \circ \Phi_{f^n(x)} \circ Df_x^n\end{aligned}$$

Note that $\Phi_y^{-1} \circ h^{cs} \circ \Phi_x(0) = 0$. Fix $r > 0$ and let $v \in E_x^u$ be any vector with $\|v\| = r$. Set $\zeta := \sup_{x \in M} \max\{\|Df|_{E^u}\|, \|Df^{-1}|_{E^u}\|\}$. Then there is an integer $n(v) \geq 0$ such that

$$\zeta^{-1} \leq \|Df^{n(v)}(v)\| \leq \zeta$$

If $w \in E_x^u$ is any other vector with $\|w\| = r$, we have by the definition of the quasiconformal distortion K^u that

$$\zeta^{-1} \cdot K^u(x, n(v))^{-1} \leq \|Df^{n(v)}(w)\| \leq \zeta \cdot K^u(x, n(v)).$$

Since f is uniformly u -quasiconformal we conclude that there is a constant $\kappa \geq 1$ such that for every $x \in M$, and every $v, w \in E_x^u$ with $\|v\| = \|w\| = r$,

$$\kappa^{-1} \leq \|Df^{n(v)}(w)\| \leq \kappa.$$

In other words, we can choose $n(v) = n(\|v\|)$ to only depend on the norm of v . For definiteness we take $n(\|v\|)$ to be the maximal integer satisfying the above inequality.

The charts $\{\Phi_x\}_{x \in M}$ are uniformly C^1 on the balls of radius κ in E_x^u as x ranges over M . By applying (3.1) in the coordinates given by $\Phi_{f^{n(\|v\|)}(x)}$ and $\Phi_{f^{n(\|v\|)}(y)}$ we conclude that there is a constant $K \geq 1$ independent of x, y and v (as long as $\|v\| = r$) such that

$$K^{-1} \leq \left\| \Phi_{f^{n(\|v\|)}(y)}^{-1} (h_{n(\|v\|)}^{cs} (\Phi_{f^{n(\|v\|)}(x)} (Df_x^{n(\|v\|)}(v)))) \right\| \leq K$$

Since the linear map $Df^{-n(\|v\|)}$ has uniformly bounded dilatation by our assumption that f is uniformly u -quasiconformal, we conclude after taking ratios that for a possibly larger constant K and any pair of vectors $v, w \in E_x^u$ with $\|v\| = \|w\| = r$,

$$\frac{\|\Phi_y^{-1}(h^{cs}(\Phi_x(v)))\|}{\|\Phi_y^{-1}(h^{cs}(\Phi_x(w)))\|} \leq K^2.$$

This holds for every $r > 0$, no matter how small r is. We thus conclude that the linear dilatation of $\Phi_y^{-1} \circ h^{cs} \circ \Phi_x$ at 0 is bounded above by K^2 for any $x \in M$ and $y \in \mathcal{W}_{loc}^{cs}(x)$.

To bound the dilatation of $\Phi_y^{-1} \circ h^{cs} \circ \Phi_x$ at points other than 0 in a ball of bounded radius centered at 0 in E_x^u , we write for $z \in \mathcal{W}_{loc}^u(x)$,

$$\Phi_y^{-1} \circ h^{cs} \circ \Phi_x = (\Phi_y^{-1} \circ \Phi_{h^{cs}(z)}) \circ (\Phi_{h^{cs}(z)}^{-1} \circ h^{cs} \circ \Phi_z) \circ (\Phi_z^{-1} \circ \Phi_x)$$

The dilatation of $\Phi_{h^{cs}(z)}^{-1} \circ h^{cs} \circ \Phi_z$ at 0 is bounded above by K^2 , by our above reasoning. By Proposition 8 the maps $\Phi_y^{-1} \circ \Phi_{h^{cs}(z)}$ and $\Phi_z^{-1} \circ \Phi_x$ are both affine maps with linear parts $H_{y h^{cs}(z)}^u$ and H_{zx}^u respectively. Since we are working on balls of bounded radius centered at 0 in E_x^u and E_y^u respectively and the unstable holonomies are linear maps depending continuously on the base points which

are thus a bounded distance from the identity, we conclude that after possibly increasing the constant K the linear dilatation of $\Phi_y^{-1} \circ h^{cs} \circ \Phi_x$ at $\Phi_x^{-1}(z)$ is bounded above by K^3 . This gives us the required quasiconformality assertion of the lemma. \square

We next recall some standard analytic properties of quasiconformal mappings. A homeomorphism $h : U \rightarrow V$ between open domains of \mathbb{R}^k is *absolutely continuous* if it preserves the collection of zero sets of k -dimensional Lebesgue measure. There is a natural Lebesgue measure class on the space of affine lines in \mathbb{R}^k given by the identification of this space with all translates of lines in \mathbb{R}^k , i.e., with $\mathbb{R}\mathbb{P}^{k-1} \times \mathbb{R}^k$. Such a homeomorphism is *absolutely continuous on lines* if for each of the coordinate directions e_1, \dots, e_k in \mathbb{R}^k we have that for almost every line $\ell \subset \mathbb{R}^k$ parallel to e_i the restriction of h to a homeomorphism $\ell \cap U \rightarrow h(\ell \cap U)$ takes subsets of $\ell \cap U$ of 1-dimensional Lebesgue measure zero to zero measure sets of $h(\ell \cap U)$, where $h(\ell \cap U)$ is equipped with the 1-dimensional Hausdorff measure in \mathbb{R}^k . Here the “almost everywhere” quantifier on the space of lines parallel to e_i (which we identify with \mathbb{R}^{k-1}) is taken with respect to the Lebesgue measure on \mathbb{R}^{k-1} . By Fubini’s theorem if h is ACL then h is absolutely continuous.

Let vol_k denote the standard Lebesgue measure on \mathbb{R}^k . For an absolutely continuous homeomorphism $h : U \rightarrow V$ we define the *Jacobian* of h to be the Radon-Nikodym derivative of $h_*(\text{vol}_k)$ with respect to vol_k and denote it by $Jac(h)$.

We let $\|\cdot\|_\infty$ denote the L^∞ norm on measurable functions $f : \mathbb{R}^k \rightarrow \mathbb{R}$,

$$\|f\|_\infty = \inf_V \sup_{x \in V} |f(x)|$$

where the infimum is taken over all measurable subsets V of \mathbb{R}^k with $\text{vol}_k(\mathbb{R}^k \setminus V) = 0$. A standard reference for the claims in Proposition 11 as well as a more precise discussion of the ACL property is Väisälä’s book [34].

PROPOSITION 11. *Suppose that $h : U \rightarrow V$ is a K -quasiconformal homeomorphism between open subsets of \mathbb{R}^k , $k \geq 2$. Then h is ACL, differentiable vol_k -a.e. in U , and we also have $\|Dh_x\|_\infty \cdot \|(Dh_x)^{-1}\|_\infty \leq K$.*

We next discuss the notion of absolute continuity of a foliation. Let m be a measure on M which is equivalent to the Riemannian volume. Let \mathcal{W} be a k -dimensional foliation of an n -dimensional Riemannian manifold M which is tangent to a continuous subbundle E of M . For each $y \in M$ we let $\mathcal{W}_r(y)$ denote the ball of radius r in the induced Riemannian metric on the leaf $\mathcal{W}(y)$ through y which is centered at y . Then there is a family of conditional measures $\{m_x^\mathcal{W}\}_{x \in M}$ of m on the foliation \mathcal{W} with the following properties: for each $x \in M$ we have $m_x^\mathcal{W}(M \setminus \mathcal{W}(x)) = 0$, the function $x \rightarrow m_x^\mathcal{W}$ is constant on the leaves of \mathcal{W} , and if S_x denotes a small $(n - k)$ -dimensional disk passing through x and transverse to \mathcal{W} and

$$V_x := \bigcup_{y \in S_x} \mathcal{W}_r(y),$$

denotes an open neighborhood of x , then up to scaling $m_y^\mathcal{W}$ on each local leaf $\mathcal{W}_r(y)$ the family $\{m_y^\mathcal{W}|_{\mathcal{W}_r(y)}\}_{y \in S_x}$ coincides with the classically defined notion of disintegration of a measure with respect to a measurable partition given by Rokhlin [28]. The family $\{m_x^\mathcal{W}\}_{x \in M}$ is uniquely defined up to m -null sets of M and up to

scaling each of the measures on a given leaf of \mathcal{W} by a positive constant. We refer to [3, Section 3] for the proof of the existence and uniqueness of the disintegration claimed in this paragraph.

For a submanifold S of M we let ν^S be the induced Riemannian volume on S from M . We define a k -dimensional foliation \mathcal{W} to be *strongly absolutely continuous* if for any pair of nearby smooth transversal $(n - k)$ -dimensional submanifolds S_1 and S_2 for \mathcal{W} the \mathcal{W} -holonomy map $h^{\mathcal{W}} : S_1 \rightarrow S_2$ is absolutely continuous with respect to the measures ν^{S_1} and ν^{S_2} , i.e., $h_*(\nu^{S_1})$ is absolutely continuous with respect to ν^{S_2} . Every C^1 foliation is strongly absolutely continuous. The most important examples of strongly absolutely continuous foliation for purposes are the stable and unstable foliations \mathcal{W}^s and \mathcal{W}^u of a partially hyperbolic diffeomorphism; strong absolute continuity of these foliations is well-known and a proof may be found in [1].

What we call “strong absolute continuity” is the notion of absolute continuity used in [3], but this notion of absolute continuity is too strong for our purposes. We define a weaker notion of absolute continuity below,

DEFINITION 12. A foliation \mathcal{W} is *absolutely continuous* if for each $x \in M$ there is an open neighborhood V of x and a strongly absolutely continuous foliation \mathcal{F} of V transverse to \mathcal{W} such that for any pair of points $y, z \in V$ the \mathcal{W} -holonomy map $h^{\mathcal{W}} : \mathcal{F}(y) \rightarrow \mathcal{F}(z)$ is absolutely continuous with respect to the induced Riemannian volumes on $\mathcal{F}(y)$ and $\mathcal{F}(z)$ respectively.

This definition is weaker because we only require the existence of a *particular* foliation \mathcal{F} transverse to \mathcal{W} for which the \mathcal{W} -holonomy maps between any pair of leaves are absolutely continuous. We emphasize that the transverse foliation \mathcal{F} need not be smooth in our definition.

Given a foliation \mathcal{W} we say that m has *Lebesgue disintegration along \mathcal{W}* if for m -a.e. $x \in M$ the conditional measure $m_x^{\mathcal{W}}$ on the leaf $\mathcal{W}(x)$ is equivalent to the induced Riemannian volume on $\mathcal{W}(x)$ from M . Our definition of absolute continuity is designed such that the following proposition is true,

PROPOSITION 13. *Let \mathcal{W} be an absolutely continuous foliation. Then m has Lebesgue disintegration along \mathcal{W} .*

Proof. Fix a point $x \in M$ and let V be an open neighborhood of x on which there is a strongly absolutely continuous foliation \mathcal{F} transverse to \mathcal{W} for which the \mathcal{W} -holonomy maps between any two \mathcal{F} -leaves are absolutely continuous. In the case that \mathcal{F} is a C^1 foliation, the proof that the conclusion of the proposition holds is given by [7, Proposition 6.2.2]. However the only property of the transversal foliation \mathcal{F} which is used in that proof is the strong absolute continuity, for completeness we give a detailed proof using only this strong absolute continuity property.

Without loss of generality we assume that V has a local product structure, i.e. for any $x', x'' \in V$, the local leaves $\mathcal{F}_{loc}(x') \cap V$ and $\mathcal{W}_{loc}(x'') \cap V$ intersect at exactly one point in V . If we denote $\mathcal{F}_{loc}(x') \cap V$, $\mathcal{W}_{loc}(x'') \cap V$ by $\mathcal{F}_V(x')$ and $\mathcal{W}_V(x'')$ respectively for any $x', x'' \in V$, then we have

$$(3.2) \quad V = \cup_{y \in \mathcal{W}_V(x)} \mathcal{F}_V(y) = \cup_{s \in \mathcal{F}_V(x)} \mathcal{W}_V(s)$$

Since \mathcal{F} is a strongly absolutely continuous foliation, there exists a positive measurable conditional density $\delta_y(\cdot)$ for $\nu^{\mathcal{W}_V(x)}$ -almost every $y \in \mathcal{W}_V(x)$ such that for

any measurable subset $A \subset V$ we have

$$(3.3) \quad m(A) = \int_{\mathcal{W}_V(x)} \int_{\mathcal{F}_V(y)} \mathbb{1}_A(y, z) \delta_y(z) d\nu^{\mathcal{F}_V(y)}(z) d\nu^{\mathcal{W}_V(x)}(y)$$

where we recall from above that ν^S denotes the induced Riemannian volume on the submanifold S .

Let $p_y(\cdot)$ denote the holonomy maps along the leaves of \mathcal{W} from $\mathcal{F}_V(x)$ to $\mathcal{F}_V(y)$, and let $q_y(\cdot)$ denote the Jacobian of p_y . We have

$$(3.4) \quad \int_{\mathcal{F}_V(y)} \mathbb{1}_A(y, z) \delta_y(z) d\nu^{\mathcal{F}_V(y)}(z) = \int_{\mathcal{F}_V(x)} \mathbb{1}_A(p_y(s)) \delta_y(p_y(s)) q_y(s) d\nu^{\mathcal{F}_V(x)}(s),$$

and by changing the order of integration in (3.3) we get

$$(3.5) \quad m(A) = \int_{\mathcal{F}_V(x)} \int_{\mathcal{W}_V(x)} \mathbb{1}_A(p_y(s)) \delta_y(p_y(s)) q_y(s) d\nu^{\mathcal{W}_V(x)}(y) d\nu^{\mathcal{F}_V(x)}(s).$$

Let $\bar{p}_s(\cdot)$ denote the holonomy map along the leaves of \mathcal{F}_V from $\mathcal{W}_V(s)$ to $\mathcal{W}_V(x)$. Since \mathcal{F} is a strongly absolutely continuous foliation the map $\bar{p}_s(\cdot)$ is absolutely continuous and thus admits a Jacobian \bar{q}_s with respect to the induced volumes on $\mathcal{W}_V(s)$ and $\mathcal{W}_V(x)$ respectively.

We transform the inner integral in (3.5) into an integral over $\mathcal{W}_V(s)$ by making the change of variables $r = p_y(s)$. Note that $y = y(r)$ is uniquely determined by r and is continuous as a function of r . Therefore we have

$$(3.6) \quad \begin{aligned} & \int_{\mathcal{W}_V(x)} \mathbb{1}_A(p_y(s)) \delta_y(p_y(s)) d\nu^{\mathcal{W}_V(x)}(y) \\ &= \int_{\mathcal{W}_V(s)} \mathbb{1}_A(r) q_{y(r)}(s) \delta_{y(r)}(r) \bar{q}_s(r) d\nu^{\mathcal{W}_V(s)}(r). \end{aligned}$$

Combining this with (3.5) we get

$$(3.7) \quad m(A) = \int_{\mathcal{F}_V(x)} \int_{\mathcal{W}_V(s)} \mathbb{1}_A(s, r) q_{y(r)}(s) \delta_{y(r)}(r) \bar{q}_s(r) d\nu^{\mathcal{W}_V(s)}(r) d\nu^{\mathcal{F}_V(x)}(s),$$

which implies the statement of the Lemma. \square

COROLLARY 14. *Let f be a C^∞ dynamically coherent partially hyperbolic diffeomorphism.*

- (1) *Suppose that f is uniformly u -quasiconformal and there exists $C > 0$ such that if $y \in \mathcal{W}_{loc}^c(x)$ then*

$$d_c(f^n(x), f^n(y)) \leq C, \text{ for every } n \geq 0.$$

Then m has Lebesgue disintegration along \mathcal{W}^{cs} -leaves.

- (2) *Suppose that f is uniformly quasiconformal and there exists $C > 0$ such that if $y \in \mathcal{W}_{loc}^c(x)$ then*

$$d_c(f^n(x), f^n(y)) \leq C, \text{ for every } n \in \mathbb{Z}.$$

Then for $$ $\in \{cs, c, cu\}$, m has Lebesgue disintegration along \mathcal{W}^* leaves.*

Proof. By Lemma 10 and Proposition 11 the hypotheses of (1) imply that the local \mathcal{W}^{cs} -holonomy maps between local leaves of the unstable foliation \mathcal{W}^u are absolutely continuous. Since the stable and unstable foliations \mathcal{W}^s and \mathcal{W}^u are strongly absolutely continuous this implies by Proposition 13 that m has Lebesgue disintegration along \mathcal{W}^{cs} leaves.

Under the hypotheses of (2) we may also apply Lemma 10 to the local cs -holonomies of f^{-1} , i.e., to the cu -holonomies of f . This implies that the local \mathcal{W}^{cu} -holonomy maps between local leaves of the unstable foliation \mathcal{W}^s are absolutely continuous with bounded Jacobians as well, and thus by Proposition 13 (using \mathcal{W}^s as the transverse strongly absolutely continuous foliation) we conclude that m also has Lebesgue disintegration along \mathcal{W}^{cu} leaves.

For each $x \in M$ the cu -leaf $\mathcal{W}^{cu}(x)$ is foliated by \mathcal{W}^u -leaves and this foliation is strongly absolutely continuous when we consider $\mathcal{W}^{cu}(x)$ to be the ambient manifold. The holonomy of the \mathcal{W}^c foliation between local \mathcal{W}^u leaves inside of $\mathcal{W}^{cu}(x)$ coincides with the \mathcal{W}^{cs} holonomy in M . Thus by Lemma 10 and Proposition 11 these local \mathcal{W}^c -holonomy maps are absolutely continuous.

Let m_x^{cu} denote the conditional volume of m on $\mathcal{W}^{cu}(x)$. Since the holonomy maps of \mathcal{W}^c between local \mathcal{W}^u leaves inside of $\mathcal{W}^{cu}(x)$ are absolutely continuous we can apply Proposition 13 again to obtain that m_x^{cu} has Lebesgue disintegration along \mathcal{W}^c leaves inside of $\mathcal{W}^{cu}(x)$. Since this holds for every $x \in M$ and m has Lebesgue disintegration along \mathcal{W}^{cu} leaves it follows that m has Lebesgue disintegration along \mathcal{W}^c leaves. \square

4. HIGHER REGULARITY OF THE CENTER FOLIATION

In this section we will prove higher regularity properties of the \mathcal{W}^c , \mathcal{W}^{cs} , and \mathcal{W}^{cu} foliations under stronger assumptions on f . Unless stated otherwise, in all of the claims of this section we assume that f is a C^∞ dynamically coherent partially hyperbolic diffeomorphism which is uniformly quasiconformal. We further assume that f preserves an invariant measure m which is smoothly equivalent to the Riemannian volume on M .

4.1. A fiber bundle construction. We first formulate an additional condition on f which is related to the proof of Theorem 4.

Suppose that $\dim E^c = 1$, E^c is orientable, $f(\mathcal{W}^c(x)) = \mathcal{W}^c(x)$ for every $x \in M$, and f has no fixed points. Each center leaf $\mathcal{W}^c(x)$ has as its universal cover a copy $\widetilde{\mathcal{W}}^c(x)$ of \mathbb{R} with orientation determined by the orientation of E^c . The restriction of f to $\mathcal{W}^c(x)$ lifts to an orientation-preserving diffeomorphism \tilde{f} of $\widetilde{\mathcal{W}}^c(x)$ with no fixed points. Fix a lift \tilde{x} of x and let \tilde{U}_x be the closed segment joining $\tilde{f}^{-1}(\tilde{x})$ to $\tilde{f}(\tilde{x})$ inside of $\widetilde{\mathcal{W}}^c(x)$. We then let U_x be the projection of this segment to $\mathcal{W}^c(x)$. An easy exercise shows that the neighborhood U_x is independent of the chosen lift of x . It is also clear that for every $x \in M$ we have $f(U_x) = U_{f(x)}$.

We say that f *does not wrap* if for each $x \in M$, the neighborhood U_x of x in $\mathcal{W}^c(x)$ is simply connected, i.e., it is a line segment instead of a circle.

For the remainder of Section 4 we will assume that f satisfies one of the following two assumptions,

- (A) \mathcal{W}^c is uniformly compact; or
- (B) $\dim E^c = 1$, $f(\mathcal{W}^c(x)) = \mathcal{W}^c(x)$ for every $x \in M$, E^c is orientable with orientation preserved by f , f has no fixed points, and f does not wrap.

In the case that f satisfies assumption (B) we set $\{U_x\}_{x \in M}$ to be the family of neighborhoods of points of M inside of the center foliation constructed above. When f satisfies assumption (A) we instead set $U_x = \mathcal{W}^c(x)$. In both cases

we have the properties that $f(U_x) = U_{f(x)}$ and there is an $R > 0$ such that $\text{diam}_c(U_x) \leq R$ for every $x \in M$.

We note that if ψ_1 is the time-1 map of an Anosov flow ψ_t which has no periodic orbits of period ≤ 2 then assumption (B) always holds for C^1 -small enough perturbations of ψ_1 . We refer to the proof of Theorem 4 at the end of this section for the details behind this assertion.

PROPOSITION 15. *Under assumptions (A) or (B), there is $R \geq r > 0$ such that for any $x \in M, y \in \mathcal{W}^c(x)$ with $d_c(x, y) \leq r$, we have*

$$d_c(f^n(x), f^n(y)) \leq R, \quad \forall n \in \mathbb{Z}$$

Proof. Under assumption (A) we may take

$$r = R = \sup_{x \in M} \text{diam}_c(\mathcal{W}^c(x)) < \infty$$

since we assume that the center foliation is uniformly compact so that there is a uniform bound on the diameters of center leaves.

Under assumption (B) we let

$$r = \inf_{x \in M} d_c(x, f(x)), \quad R = 2 \sup_{x \in M} d_c(x, f(x)).$$

Since M is compact and f has no fixed points we have $0 < r \leq R/2 < \infty$. If $y \in \mathcal{W}^c(x)$ satisfies $d_c(x, y) \leq r$ then $y \in U_x$ and we conclude that $f^n(y) \in U_{f^n(x)}$ for every $n \in \mathbb{Z}$. Since $\text{diam}_c(U_x) \leq R$ for every $x \in M$, the statement of the Proposition follows. \square

As a consequence of Proposition 15, all of the work from Section 3 applies to both f and f^{-1} for systems satisfying assumptions (A) or (B). In particular the center-stable holonomy maps $h_{xy}^{cs} : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y)$ and center-unstable holonomy maps $h_{xz}^{cu} : \mathcal{W}_{loc}^s(x) \rightarrow \mathcal{W}_{loc}^s(z)$ for $x \in M, y \in \mathcal{W}_{loc}^{cs}(x), z \in \mathcal{W}_{loc}^{cu}(x)$ are all K -quasiconformal for some constant $K \geq 1$. Consequently m has Lebesgue disintegration along each of the foliations $\mathcal{W}^{cs}, \mathcal{W}^{cu}$, and \mathcal{W}^c by Corollary 14. By combining this with the strong absolute continuity of the foliations \mathcal{W}^u and \mathcal{W}^s we conclude that m has Lebesgue disintegration along all of the invariant foliations \mathcal{W}^* for f .

We now consider the space

$$\mathcal{E} = \{(x, y) \in M^2 : y \in U_x\},$$

and we define $F : \mathcal{E} \rightarrow \mathcal{E}$ by $F(x, y) = (f(x), f(y))$.

PROPOSITION 16. *\mathcal{E} is a continuous fiber bundle over M with compact fibers. F preserves an invariant measure μ on \mathcal{E} which locally decomposes as the product of the volume m on M and the conditional volume m_x^c on U_x .*

Proof. We first show that for $r > 0$ sufficiently small and $y \in \mathcal{W}_r^u(x)$, under either assumption (A) or (B), the unstable holonomy map h_{xy}^u from $\mathcal{W}_{loc}^c(x)$ to $\mathcal{W}_{loc}^c(y)$ can be extended to a homeomorphism $h^u : U_x \rightarrow U_y$. Under either assumption (A) or (B) the neighborhoods $U_x \subset \mathcal{W}^c(x)$ are uniformly compact (i.e., have d_c -diameter uniformly bounded in x) and depend continuously on x in the Hausdorff topology on sets. Thus we can choose r small enough that for any $x \in M, y \in \mathcal{W}_r^u(x)$ we have that $\mathcal{W}_{loc}^u(z) \cap U_y$ consists of at most one point for any $z \in U_x$.

Under assumption (B) this last assertion requires the condition that f does not wrap.

We claim that there is in fact exactly one point in $\mathcal{W}_{loc}^u(z) \cap U_y$. This is obvious under assumption (A) since $\mathcal{W}_{loc}^u(z)$ must intersect $\mathcal{W}^c(y)$ if y is close enough to x , by the uniform compactness of the center foliation. Under assumption (B) let $\varepsilon > 0$ be a constant chosen small enough that for each $x \in M$ the ε -neighborhood U_x^ε of U_x inside of $\mathcal{W}^c(x)$ still satisfies the no wrapping condition, that is to say, U_x^ε remains an interval instead of a circle. Then for $\varepsilon > 0$ sufficiently small, r sufficiently small depending of ε , and any $x \in M$, $y \in \mathcal{W}_r^u(x)$ and $z \in U_x$ the intersection $\mathcal{W}_{loc}^u(z) \cap U_y^\varepsilon$ consists of exactly one point. Thus $h^u : U_x \rightarrow U_y^\varepsilon$ is an orientation-preserving homeomorphism onto its image inside of U_y^ε . But if $y \in \mathcal{W}_r^u(x)$ for r sufficiently small then $f(y)$ and $f^{-1}(y)$ lie in $\mathcal{W}_{loc}^u(f(x))$ and $\mathcal{W}_{loc}^u(f^{-1}(x))$ respectively. Thus the endpoints of the interval which is the image of U_x under h^u are $f^{-1}(y)$ and $f(y)$. This shows that h^u actually gives a homeomorphism from U_x to U_y .

Hence $h^u : U_x \rightarrow U_y$ is a homeomorphism for $y \in \mathcal{W}_r^u(x)$. By similar reasoning (possibly taking r smaller) for any $x \in M$, $y \in \mathcal{W}_r^s(x)$ the stable holonomy $h^s : U_x \rightarrow U_y$ is also a homeomorphism. Finally, it is easy to see that for any $y \in \mathcal{W}^c(x)$ there is a homeomorphism $h^c : U_x \rightarrow U_y$ depending uniformly continuously on the pair (x, y) : in the case of assumption (A) this is trivial since $U_x = U_y$. In the case of assumption (B) each of the subsets U_y of $\mathcal{W}^c(x)$ is an interval determined canonically by its endpoints $f^{-1}(y)$ and $f(y)$ according to the construction at the beginning of this section. These endpoints depend continuously on $y \in \mathcal{W}^c(x)$ hence it follows that we can find a continuous family of orientation-preserving homeomorphisms $h^c : U_y \rightarrow U_x$ identifying these intervals for y near x . Putting all of this together, for any z close enough to x we can find a composition of three homeomorphisms

$$U_z \rightarrow U_{h^s(z)} \rightarrow U_{h^u(h^s(z))} \rightarrow U_{h^c(h^u(h^s(z)))} = U_x$$

which depends continuously on z , where $h^s(z) = \mathcal{W}_{loc}^s(z) \cap \mathcal{W}_{loc}^{cu}(x)$, $h^u(h^s(z)) = \mathcal{W}_{loc}^u(h^s(z)) \cap \mathcal{W}_{loc}^c(x)$. This proves that \mathcal{E} is a continuous fiber bundle over M with compact fibers.

We now prove the second assertion. Under assumption (A) we consider the measurable partition of M into compact center fibers $M = \bigcup_{x \in M} \mathcal{W}^c(x)$ and let $\{m_x^c\}_{x \in M}$ be the family of conditional measures of m on the center fibers $\mathcal{W}^c(x)$ determined by this partition. Since f preserves m we have $f_* m_x^c = m_{f(x)}^c$ for m -a.e. $x \in M$.

In the case of assumption (B) we refer to [3, Section 3]. It is shown there that for the foliation \mathcal{W}^c of M there is a measurable family of conditional measures $\{\hat{m}_x^c\}_{x \in M}$ supported on the leaves $\mathcal{W}^c(x)$ of the center foliation such that for $y \in \mathcal{W}^c(x)$ the measures \hat{m}_x^c and \hat{m}_y^c coincide up to a constant factor. We normalize these measures such that $\hat{m}_x^c(U_x) = 1$. Furthermore, since f fixes all of the leaves of \mathcal{W}^c , by [3, Proposition 3.3] we have $f_* \hat{m}_x^c = \hat{m}_x^c$ for m -a.e. $x \in M$. We define m_x^c to be the restriction of \hat{m}_x^c to U_x . Since $f(U_x) = U_{f(x)}$ we conclude that $f_* m_x^c$ is a constant multiple of $m_{f(x)}^c$, and by our choice of normalization of the measures \hat{m}_x^c this implies $f_* m_x^c = m_{f(x)}^c$ since these measures both assign mass 1 to $U_{f(x)}$.

The measures m_x^c are equivalent to the Riemannian volume on U_x since m has Lebesgue disintegration along \mathcal{W}^c leaves. We define the measure μ on the fiber bundle \mathcal{E} by setting, for any measurable set $A \subset \mathcal{E}$,

$$\mu(A) = \int \mathbb{1}_A(x, y) dm_x^c(y) dm(x),$$

where $\mathbb{1}_A$ denotes the characteristic function of A . Since f preserves m and $f_* m_x^c = m_{f(x)}^c$ for m -a.e. $x \in M$, we conclude that μ is F -invariant. \square

4.2. Conformal structures. We now introduce the bundle $\mathcal{C}E^u$ of conformal structures on E^u over M . For more details related to the discussion that follows we defer to [21]. The fiber $\mathcal{C}E_x^u$ over x is the space of all inner products on E_x^u modulo scaling by a nonzero real number, which can be identified with the nonpositively curved Riemannian symmetric space $SL(k, \mathbb{R})/SO(k, \mathbb{R})$. Each fiber thus carries a canonical Riemannian metric ρ_x given by an isometric identification of $\mathcal{C}E_x^u$ with $SL(k, \mathbb{R})/SO(k, \mathbb{R})$. We will always explicitly identify $\mathcal{C}E_x^u$ with the space of inner products on E_x^u for which the determinant of a positively oriented orthonormal basis is 1 in the reference inner product on E_x^u induced from the given Riemannian metric on TM .

Any linear isomorphism $A : E_x^u \rightarrow E_y^u$ induces a map $A^* : \mathcal{C}E_y^u \rightarrow \mathcal{C}E_x^u$ by, for $\tau_x \in \mathcal{C}E_x^u$ and any $v, w \in E_y^u$,

$$A^* \tau_x(v, w) = \frac{\tau_x(A(v), A(w))}{\det(A)^{2/k}},$$

where we recall that $k = \dim E^u$ and $\det(A)$ denotes the determinant of A in the metric induced from TM . The induced map A^* is an isometry from $(\mathcal{C}E_y^u, \rho_y)$ to $(\mathcal{C}E_x^u, \rho_x)$.

A (measurable) *conformal structure* on E^u is a measurable section $\tau : M \rightarrow \mathcal{C}E^u$ defined on the complement of an m -null set of M . We say that a measurable conformal structure is *invariant* if $Df_x^* \tau_{f(x)} = \tau_x$ for m -a.e. $x \in M$. A measurable conformal structure is *bounded* if there is a constant $C > 0$ such that $\rho_x(\tau_x, \tau_x^0) \leq C$ for m -a.e. $x \in M$, where τ^0 denotes the conformal structure on E^u induced from the Riemannian metric on TM . The condition that f is uniformly u -quasiconformal is equivalent to the existence of a constant $C > 0$ such that for every $x \in M$,

$$\rho_x \left((Df_x^n)^* \tau_{f^n(x)}^0, \tau_x^0 \right) \leq C \quad \forall n \in \mathbb{Z}.$$

The following measure-theoretic lemma is vital for recovering holonomy invariance properties of measurable objects by guaranteeing simultaneous recurrence to a continuity set on a full measure set of points. Our first application will be to show that a measurable invariant conformal structure for f must be invariant under the stable and unstable holonomies H^s and H^u on a full measure subset of M .

LEMMA 17. *Let T be a measure-preserving transformation of a finite measure space (X, μ) and let $\{K_n\}_{n \geq 1}$ be a sequence of measurable subsets of X with $\sum_{n=1}^{\infty} \mu(X \setminus K_n) < \infty$. Then there is a full measure subset $\Omega \subseteq X$ with the property that if $x, y \in \Omega$ then there is an $n \in \mathbb{N}$ and a sequence $n_k \rightarrow \infty$ with $T^{n_k}(x) \in K_n, T^{n_k}(y) \in K_n$ for each n_k .*

Proof. By the Birkhoff ergodic theorem, for each $n \in \mathbb{N}$ the Birkhoff averages $\frac{1}{k} \sum_{j=0}^{k-1} \mathbb{1}_{K_n}(T^j(x))$ converge pointwise (on a measurable set E_n with $\mu(X \setminus E_n) = 0$) as $k \rightarrow \infty$ to a nonnegative T -invariant measurable function P_n with integral $\mu(K_n)$. Define $\Omega \subset X$ by

$$\Omega = \bigcap_{n=1}^{\infty} E_n \cap \left\{ x \in X : \exists N \in \mathbb{N} \text{ such that for } n \geq N, P_n(x) > \frac{3}{4} \right\}.$$

We claim that $\mu(X \setminus \Omega) = 0$. Consider the sets

$$B_n = \left\{ x \in E_n : 1 - P_n(x) \geq \frac{1}{4} \right\}.$$

By the Markov inequality we have

$$\mu(B_n) \leq 4 \int_X 1 - P_n d\mu = 4\mu(X \setminus K_n)$$

Since $\sum_{n=1}^{\infty} \mu(X \setminus K_n) < \infty$ by hypothesis, we conclude by the Borel-Cantelli lemma that for μ -a.e. $x \in X$ there are only finitely many n such that $x \in B_n$. This implies that Ω is a full measure subset of X .

Now we verify that Ω has the desired properties of the Lemma's conclusion. If $x, y \in \Omega$ are any two given points then from the definition of Ω there is a common n such that $P_n(x) > \frac{3}{4}$ and $P_n(y) > \frac{3}{4}$. We explain how to construct n_k inductively from n_{k-1} , with our construction also showing how to construct the initial n_1 from $n_0 = 0$. Given n_{k-1} , we choose $N_k > n_{k-1}$ large enough that

$$\frac{1}{N_k} \sum_{j=n_{k-1}}^{N_k} \mathbb{1}_{K_n}(T^j(x)) > \frac{3}{4}$$

and the same for y , and we also take N_k large enough that $\frac{N_k - n_{k-1}}{N_k} > \frac{7}{8}$. The existence of this N_k is guaranteed by the fact that $P_n(x) > \frac{3}{4}$. We conclude from these estimates that

$$\frac{|\{j \in [n_{k-1}, N_k] : T^j(x) \in K_n\}|}{N_k - n_{k-1}} > \frac{4}{7}$$

and the same for y , where $[n_{k-1}, N_k]$ denotes the set of integers j satisfying $n_{k-1} \leq j \leq N_k$. It follows that there is a common $n_k \in [n_{k-1}, N_k]$ such that $T^{n_k}(x) \in K_n$ and $T^{n_k}(y) \in K_n$. Inducting on k completes the proof. \square

Proposition 18 below summarizes the essential properties of invariant conformal structures for uniformly quasiconformal linear cocycles which we will need. It is a slight improvement of the proof of [22, Proposition 4.4] as it removes the assumption of ergodicity of f with respect to the volume m which was used in that proof.

PROPOSITION 18. *Suppose that f is uniformly u -quasiconformal and volume-preserving. Then there is an invariant bounded measurable conformal structure $\tau : M \rightarrow CE^u$. Furthermore there is a full measure subset Ω of M such that if $x, y \in \Omega$ and $y \in \mathcal{W}_{loc}^u(x)$ then*

$$(H_{xy}^u)^* \tau_y = \tau_x,$$

and similarly if $y \in \mathcal{W}_{loc}^s(x)$ then

$$(H_{xy}^s)^* \tau_y = \tau_x.$$

Proof. By [21, Proposition 2.4] the uniform quasiconformality of the linear cocycle $Df|_{E^u}$ implies that there is an invariant bounded measurable conformal structure $\tau : M \rightarrow CE^u$.

By Lusin's theorem we can find an increasing sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of M such that τ is uniformly continuous on K_n and $m(M \setminus K_n) < 2^{-n}$. Let $\Omega \subset M$ be the full measure set of points satisfying the conclusion of Lemma 17 for both f and f^{-1} . We also require that τ is defined and Df -invariant on Ω .

Let $x, y \in \Omega$ be given with $y \in \mathcal{W}_{loc}^s(x)$. Then there is an $n > 0$ and a sequence $n_k \rightarrow \infty$ such that $f^{n_k}(x)$ and $f^{n_k}(y)$ both lie in K_n for each n_k . Then

$$\begin{aligned} \rho_x(\tau_x, (H_{xy}^s)^* \tau_y) &= \rho_x((Df_x^{n_k})^* \tau_{f^{n_k}(x)}, (H_{xy}^u)^* (Df_y^{n_k})^* \tau_{f^{n_k}(y)}) \\ &= \rho_x((Df_x^{n_k})^* \tau_{f^{n_k}(x)}, (Df_x^{n_k})^* (H_{f^{n_k}x f^{n_k}y}^s)^* \tau_{f^{n_k}(y)}) \\ &= \rho_{f^{n_k}(x)}(\tau_{f^{n_k}(x)}, (H_{f^{n_k}x f^{n_k}y}^s)^* \tau_{f^{n_k}(y)}) \\ &\leq \rho_{f^{n_k}(x)}(\tau_{f^{n_k}(x)}, (I_{f^{n_k}x f^{n_k}y})^* \tau_{f^{n_k}(y)}) \\ &\quad + \rho_{f^{n_k}(x)}((I_{f^{n_k}x f^{n_k}y})^* \tau_{f^{n_k}(y)}, (H_{f^{n_k}x f^{n_k}y}^s)^* \tau_{f^{n_k}(y)}), \end{aligned}$$

where we recall that $I_{xy} : E_x^u \rightarrow E_y^u$ is our chosen Hölder continuous family of identifications of nearby fibers of E^u . The uniform continuity of τ on K_n implies that

$$\rho_{f^{n_k}(x)}(\tau_{f^{n_k}(x)}, (I_{f^{n_k}x f^{n_k}y})^* \tau_{f^{n_k}(y)}) \rightarrow 0,$$

as $n_k \rightarrow \infty$ since $d(f^{n_k}(x), f^{n_k}(y)) \rightarrow 0$ as $n_k \rightarrow \infty$. Since $\|I_{f^{n_k}x f^{n_k}y} - H_{f^{n_k}x f^{n_k}y}^s\| \rightarrow 0$ uniformly as $n_k \rightarrow \infty$ we also conclude that

$$\rho_{f^{n_k}(x)}((I_{f^{n_k}x f^{n_k}y})^* \tau_{f^{n_k}(y)}, (H_{f^{n_k}x f^{n_k}y}^s)^* \tau_{f^{n_k}(y)}) \rightarrow 0,$$

as $n_k \rightarrow \infty$. Combining these two facts gives $\tau_x = (H_{xy}^s)^* \tau_y$.

The same proof replacing \mathcal{W}^s by \mathcal{W}^u and n_k by $-n_k$ shows that if $x, y \in \Omega$ with $y \in \mathcal{W}_{loc}^u(x)$ then $\tau_x = (H_{xy}^u)^* \tau_y$. \square

4.3. Equivariance properties of the center holonomy. In the next proposition we write $h_{xy}^c : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y)$ for the center-stable holonomy between local unstable leaves inside the same center-unstable leaf, which coincides with the center holonomy between these leaves.

PROPOSITION 19. *Let*

$$\begin{aligned} Q = \{x \in M : \text{for } m_x^c\text{-a.e. } y \in U_x, \\ h_{xy}^c : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y) \text{ is differentiable at } x\}. \end{aligned}$$

Then $m(Q) = 1$.

Proof. For $x \in M$ we let m_x^u denote the conditional measure of m on $\mathcal{W}_{loc}^u(x)$. Let m_x^{cu} be the conditional measure of m on the subset

$$S_x = \bigcup_{y \in \mathcal{W}_{loc}^u(x)} U_y \subset \mathcal{W}_{loc}^{cu}(x).$$

Since m has Lebesgue disintegration along each of the foliations \mathcal{W}^u , \mathcal{W}^c , and \mathcal{W}^{cu} , for m -a.e. $x \in M$ the measure m_x^{cu} decomposes as conditional measures in two different ways,

$$m_x^{cu} \asymp \int_{U_x} m_y^u dm_x^c(y) \asymp \int_{\mathcal{W}_{loc}^u(x)} m_y^c dm_x^u(y)$$

where we use the notation \asymp to indicate that the two measures are equivalent on S_x . By Lemma 10 and Proposition 11, for every $y \in U_x$ the center holonomy map $h_{yx}^c : \mathcal{W}_{loc}^u(y) \rightarrow \mathcal{W}_{loc}^u(x)$ is differentiable at m_y^u -a.e. $z \in \mathcal{W}_{loc}^u(y)$. Thus if we set

$$T_x = \{z \in S_x : h_{xy}^c \text{ is differentiable at } z \text{ for } y = h_{zx}^u(z)\},$$

then by the first expression for m_x^{cu} we have $m_x^{cu}(T_x) = m_x^{cu}(S_x)$ for m -a.e. $x \in M$. Since T_x has full m_x^{cu} measure in S_x we conclude by the second expression for m_x^{cu} that $m_y^c(T_x \cap U_y) = m_y^c(U_y)$ for m_x^u -a.e. $y \in \mathcal{W}_{loc}^u(x)$. This immediately implies that $m_x^u(Q \cap \mathcal{W}_{loc}^u(x)) = m_x^u(\mathcal{W}_{loc}^u(x))$ from the definition of Q . Since the \mathcal{W}^u foliation is absolutely continuous and this holds for m -a.e. $x \in M$ we conclude that $m(Q) = 1$, i.e., Q has full volume in M . \square

We then define $\mathcal{Q} = \{(x, y) \in \mathcal{E} : x \in Q\}$. From the definition of Q and Proposition 19 we see that \mathcal{Q} has full μ -measure inside of \mathcal{E} . For $(x, y) \in \mathcal{Q}$ we can then define $H_{xy}^c : E_x^u \rightarrow E_y^u$ to be the derivative of h_{xy}^c at x . The map $(x, y) \rightarrow H_{xy}^c$ is clearly measurable and defined μ -a.e. by Proposition 19. Our next goal is to show that the maps H^c are equivariant with respect to the stable and unstable holonomies H^s and H^u of $Df|_{E^u}$.

LEMMA 20. *There is a full μ -measure subset Ω of \mathcal{Q} such that if $(x, y), (z, w) \in \Omega$ with $z \in \mathcal{W}_{loc}^u(x)$ and $w \in \mathcal{W}_{loc}^u(y)$ then the following equation holds,*

$$(4.1) \quad H_{zw}^c \circ H_{xz}^u = H_{yw}^u \circ H_{xy}^c,$$

and similarly if $(z, w) \in \Omega$ with $z \in \mathcal{W}_{loc}^s(x)$ and $w \in \mathcal{W}_{loc}^s(y)$ then,

$$(4.2) \quad H_{zw}^c \circ H_{xz}^s = H_{yw}^s \circ H_{xy}^c.$$

Proof. We let $\Lambda \subset M$ be the full m -measure set of points on which the invariant bounded measurable conformal structure $\tau : M \rightarrow \mathcal{C}E^u$ of Proposition 18 is defined and invariant under both Df and the stable and unstable holonomies H^s and H^u . We let $\Omega_0 \subset \mathcal{E}$ be the set of $(x, y) \in \mathcal{E}$ such that both x and y are in Λ . The absolute continuity of \mathcal{W}^c together with the construction of the measure μ implies that $\mu(\mathcal{E} \setminus \Omega_0) = 0$.

By Lusin's theorem we can find an increasing sequence of compact subsets $K_n \subset \mathcal{E}$ such that $\mu(\mathcal{E} \setminus K_n) < 2^{-n}$ and such that H^c restricts to a uniformly continuous function on each K_n . Since μ is F -invariant, by applying Lemma 17 to both F and F^{-1} there is a measurable set Ω with $\mu(\mathcal{E} \setminus \Omega) = 0$ and such that for any pair of points $(x, y), (z, w) \in \Omega$ there is an $n \in \mathbb{N}$ and a pair of infinite sequences $n_k \rightarrow \infty$ and $n'_k \rightarrow \infty$ with $F^{-n_k}(x, y), F^{-n_k}(z, w) \in K_n$ for each n_k and $F^{n'_k}(x, y), F^{n'_k}(z, w) \in K_n$.

We now prove that equation (4.1) holds. The proof for equation (4.2) will be completely analogous. Let $(x, y), (z, w) \in \Omega$ be such that $z \in \mathcal{W}_{loc}^u(x)$ and $w \in \mathcal{W}_{loc}^u(y)$. Since H^c is uniformly continuous on K_n and $d(f^{-n}x, f^{-n}z), d(f^{-n}y, f^{-n}w) \rightarrow$

0 as $n \rightarrow \infty$, we conclude that

$$(4.3) \quad \left\| H_{f^{-n_k}y f^{-n_k}w}^u \circ H_{f^{-n_k}x f^{-n_k}y}^c - H_{f^{-n_k}z f^{-n_k}w}^c \right\| \rightarrow 0,$$

as $k \rightarrow \infty$, where $\{n_k\}$ is the infinite sequence from the previous paragraph corresponding to the pair $(x, y), (z, w)$.

For $x \in \Lambda$ we let SE_x^u denote the unit sphere in E_x^u in the metric τ_x . For any two points $x, y \in \Lambda$ and an invertible linear map $A : E_x^u \rightarrow E_y^u$ we then define

$$SA(v) = \frac{A(v)}{\det(A)^{\frac{1}{k}}},$$

where the determinant is taken with respect to the induced Riemannian metric τ^0 on E^u from TM . We remark that if $A^* \tau_y = \tau_x$ then SA maps SE_x^u to SE_y^u and consequently SA is an isometry from E_x^u to E_y^u when these are given the metrics τ_x and τ_y respectively. This is because A then maps SE_x^u to $\det_\tau(A)^{\frac{1}{m}} \cdot SE_y^u$, where \det_τ denotes the determinant of this linear map with respect to the family of inner products on E^u given by τ . Our convention for representing elements of CE_x^u is to take the inner product which has determinant 1 with respect to the background metric τ_x^0 . Thus $\det_\tau = \det$ for linear maps between fibers of E^u .

We also note that A is clearly determined by SA and $\det(A)$.

For $(x, y), (z, w) \in \Omega$ given as in the statement of the lemma we will show that

$$(4.4) \quad \det(H_{zw}^c) \det(H_{xz}^u) = \det(H_{yw}^u) \det(H_{xy}^c),$$

$$(4.5) \quad SH_{zw}^c \circ SH_{xz}^u = SH_{yw}^u \circ SH_{xy}^c.$$

The desired statement of the lemma follows from these two equations.

From differentiating the equation

$$f^{-k} \circ h_{xy}^c = h_{f^{-k}x f^{-k}y}^c \circ f^{-k},$$

expressing the equivariance of center holonomy with respect to the dynamics f we obtain the equation

$$(4.6) \quad Df_y^{-k} \circ H_{xy}^c = H_{f^{-k}x f^{-k}y}^c \circ Df_x^{-k},$$

which is valid for any $(x, y) \in \mathcal{Q}$. Taking determinants and rearranging, we conclude that

$$\frac{\det(Df_y^{-k}|_{E^u})}{\det(Df_x^{-k}|_{E^u})} \det(H_{xy}^c) = \det(H_{f^{-k}x f^{-k}y}^c)$$

Applying the same equation to (z, w) with H_{zw}^c and then taking ratios at the iterates n_k gives

$$\frac{\det(Df_y^{-n_k}|_{E^u})}{\det(Df_w^{-n_k}|_{E^u})} \cdot \frac{\det(Df_z^{-n_k}|_{E^u})}{\det(Df_x^{-n_k}|_{E^u})} \cdot \frac{\det(H_{xy}^c)}{\det(H_{zw}^c)} = \frac{\det(H_{f^{-n_k}x f^{-n_k}y}^c)}{\det(H_{f^{-n_k}z f^{-n_k}w}^c)}$$

As $k \rightarrow \infty$ the right side converges to 1 by equation 4.3, the first factor in the product on the left side converges to $\det(H_{yw}^u)$, and the second factor converges to $\det(H_{xz}^u)^{-1}$. Rearranging the resulting equation gives equation (4.4).

For the second equation we first consider the following lemma which only uses the hypothesis that f is uniformly u -quasiconformal. We let $\|\cdot\|_\tau$ denote the norm on E_x^u induced by the inner product τ_x .

LEMMA 21. Suppose $x, y \in \Lambda$, $y \in \mathcal{W}_{loc}^u(x)$ and $v \in E_x^u, v' \in E_y^u$. If

$$\liminf_{n \rightarrow \infty} \|SDf^{-n}(v) - SH_{f^{-n}yf^{-n}x}^u(SDf^{-n}(v'))\|_\tau = 0,$$

then $SH_{xy}^u(v) = v'$.

Proof. Let $w = SH_{xy}^u(v)$. Then we have

$$SDf^{-n}(v) = SH_{f^{-n}yf^{-n}x}^u(SDf^{-n}(w)),$$

by the equivariance properties of the unstable holonomy H^u . Therefore we have

$$\liminf_{n \rightarrow \infty} \|SH_{f^{-n}yf^{-n}x}^u(SDf^{-n}(w)) - SH_{f^{-n}yf^{-n}x}^u(SDf^{-n}(v'))\|_\tau = 0.$$

But the invariance of τ under the unstable holonomy H^u on Λ and its invariance under Df imply that SDf^{-n} and SH^u are both isometries with respect to the family of metrics given by τ on the fibers E_x^u of the vector bundle E^u . This then implies that

$$\liminf_{n \rightarrow \infty} \|w - v'\|_\tau = 0,$$

which means that $w = v'$ as desired. \square

Now let $(x, y), (z, w) \in \Omega$ be given as in the statement of the Lemma. From equation 4.3 and the equivariance properties of H^u we conclude that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|SH_{f^{-n_k}zf^{-n_k}w}^c(SDf_z^{-n_k}(SH_{xz}^u(v))) \\ & - SH_{f^{-n_k}yf^{-n_k}w}^u(SH_{f^{-n_k}xf^{-n_k}y}^c(SDf_x^{-n_k}(v)))\| = 0. \end{aligned}$$

Applying the equivariance relation (4.6), this implies that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|SDf_w^{-n_k}(SH_{zw}^c(SH_{xz}^u(v))) \\ & - SH_{f^{-n_k}yf^{-n_k}w}^u(SDf_y^{-n_k}(SH_{xy}^c(v)))\| = 0 \end{aligned}$$

Since the measurable conformal structure τ is bounded, the norms $\|\cdot\|_\tau$ are uniformly comparable to the norm $\|\cdot\| = \|\cdot\|_{\tau_0}$ and thus this equation also holds with $\|\cdot\|$ replaced by $\|\cdot\|_\tau$. We thus conclude by Lemma 21 that

$$SH_{zw}^c(SH_{xz}^u(v)) = SH_{yw}^u(SH_{xy}^c(v)).$$

Since this holds for every $v \in E_x^u$ we deduce equation (4.1) as desired.

To prove the second equation (4.2) where instead $z \in \mathcal{W}_{loc}^s(x)$ and $w \in \mathcal{W}_{loc}^s(y)$, we follow the exact same proof, replacing H^u by H^s and $-n_k$ by n_k everywhere. \square

We next recall the following elementary lemma from analysis,

LEMMA 22. Suppose $f : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is ACL and that there is a continuous map $G : \mathbb{R}^k \rightarrow GL(k, \mathbb{R})$ such that $Df = G$ almost everywhere. Then f is a C^1 map and $Df = G$ everywhere.

Proof. Let $f = (f_1, \dots, f_k)$, $f_i : \mathbb{R}^k \rightarrow \mathbb{R}$. Since f is ACL each coordinate function f_i is ACL. Thus there exists a full Lebesgue measure set $\Lambda \subset \mathbb{R}^k$ such that for every $x \in \Lambda$ and $1 \leq i, j \leq k$, $f_i|_{x+\mathbb{R} \cdot e_j}$ is absolutely continuous and $Df = G$ for almost every point (with respect to arc length) in $\{x + \mathbb{R} \cdot e_j\}$, where e_1, \dots, e_k denote the standard basis of \mathbb{R}^k .

Absolute continuity of $f_i|_{x+\mathbb{R}\cdot e_j}$ implies that

$$\begin{aligned} f_i(x+t\cdot e_j) &= f_i(x) + \int_0^t \frac{\partial f_i}{\partial x_j}(x+s\cdot e_i) ds \\ &= f_i(x) + \int_0^t G_{ij}(x+s\cdot e_i) ds, \end{aligned}$$

where $G = (G_{ij})_{1 \leq i,j \leq k}$ is the matrix representation of G in the standard basis of \mathbb{R}^k . Since both f and G are continuous this last equation holds for all $x \in \mathbb{R}^k$ and $t \in \mathbb{R}$. This proves that for each $1 \leq i, j \leq k$ the partial derivative $\frac{\partial f_i}{\partial x_j}$ of f exists and coincides with G_{ij} . In particular all partial derivatives of f exist and are continuous at every point in \mathbb{R}^k which implies that f is C^1 and $Df = G$. \square

LEMMA 23. *For any $x \in M$ and $y \in U_x$ we have the equality*

$$\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x = H_{xy}^c$$

as maps from E_x^u to E_y^u . The measurable function H^c on Ω therefore admits a continuous extension to \mathcal{E} and the center holonomy is linear in the charts $\{\Phi_x\}_{x \in M}$.

Proof. We first consider pairs $(x, y) \in \Omega$. Since $D_0\Phi_x = Id_{E_x^u}$ for every $x \in M$ and $h_{xy}^c(x) = y$, the equation

$$D_0(\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x) = H_{xy}^c,$$

holds for any $(x, y) \in \Omega$. To compute the derivative at other points of E_x^u , we let $v \in E_x^u$, $z = \Phi_x(v)$, and $w = h_{xy}^c(z) = h_{zw}^c(z)$. We suppose that $(z, w) \in \Omega$ and compute,

$$D_v(\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x) = D_0(\Phi_y^{-1} \circ \Phi_w) \circ D_0(\Phi_w^{-1} \circ h_{zw}^c \circ \Phi_z) \circ D_v(\Phi_z^{-1} \circ \Phi_x).$$

By Proposition 8 we know that $D_0(\Phi_y^{-1} \circ \Phi_w) = H_{wy}^u$ and $D_v(\Phi_z^{-1} \circ \Phi_x) = H_{xz}^u$. Hence

$$D_v(\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x) = H_{wy}^u \circ H_{zw}^c \circ H_{xz}^u = H_{xy}^c,$$

whenever $(x, y), (z, w) \in \Omega$, by Lemma 20. Since Ω has full μ -measure we conclude that for m -a.e. $x \in M$ and m_x^c -a.e. $y \in U_x$ the map $\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x : E_x^u \rightarrow E_y^u$ is differentiable almost everywhere on E_x^u with derivative H_{xy}^c almost everywhere. By Lemma 10 the map $\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x$ is quasiconformal and therefore ACL. By Lemma 22 this implies that $\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x$ is a C^1 map with derivative H_{xy}^c everywhere, i.e., $\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x$ coincides exactly with the linear map H_{xy}^c .

Since Ω has full μ -measure in \mathcal{E} and μ is fully supported we conclude that Ω is dense in \mathcal{E} and thus the equation

$$(4.7) \quad \Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x = H_{xy}^c$$

holds on E_x^u for a dense set of pairs $(x, y) \in \mathcal{E}$. But the left side of this equation depends uniformly continuously on the pair (x, y) on as a map $\Phi_x^{-1}(\mathcal{W}_{loc}^u(x)) \rightarrow \Phi_y^{-1}(\mathcal{W}_{loc}^u(y))$ between neighborhoods of 0 in E_x^u and E_y^u of uniform size. Furthermore the linear map H_{xy}^c is determined by its restriction to a map between these neighborhoods. Hence we conclude that H_{xy}^c also depends uniformly continuously on the pairs $(x, y) \in \Omega$.

When y is close to x the map $\Phi_y^{-1} \circ h_{xy}^c \circ \Phi_x$ is uniformly close to the linear identifications $I_{xy} : E_x^u \rightarrow E_y^u$ introduced in Section 3. Hence H_{xy}^c is uniformly close to I_{xy} for $(x, y) \in \Omega$. In particular for $(x, y) \in \Omega$ the maps H_{xy}^c belong to a uniformly bounded subset of the space of invertible linear maps $E_x^u \rightarrow E_y^u$. This shows that H^c admits a continuous extension to \mathcal{E} such that equation (4.7) still holds on a neighborhood of 0 in E_x^u for any $(x, y) \in \mathcal{E}$. Finally, because Φ_x , h_{xy}^c , and H_{xy}^c all have the proper equivariance properties with respect to f^{-1} and Df^{-1} which uniformly contract E^u it follows that equation 4.7 actually holds on all of E_x^u . \square

As a corollary of Lemma 23 we deduce that equations (4.1) and (4.2) from Lemma 20 actually hold on all of \mathcal{E} because the uniform continuity of H^c implies each side of these equations is uniformly continuous in the quadruple of points x, y, z, w and both of these equations hold on a dense subset of \mathcal{E} .

Given $x \in M$, $y \in \mathcal{W}_{loc}^{cs}(x)$, we let z be the unique intersection point of $\mathcal{W}_{loc}^c(x)$ with $\mathcal{W}_{loc}^s(y)$ and define

$$H_{xy}^{cs} = H_{zy}^s \circ H_{xz}^c.$$

By the observation in the previous paragraph, if we let w be the intersection of $\mathcal{W}_{loc}^s(x)$ with $\mathcal{W}_{loc}^c(y)$ then we also have the equality

$$H_{xy}^{cs} = H_{wy}^c \circ H_{xw}^s.$$

We note that H_{xy}^{cs} depends in a uniformly continuous fashion x and y from the uniform continuity of H^c and H^s .

LEMMA 24. *The center-stable holonomy $h_{xy}^{cs} : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y)$ between two local unstable leaves is C^1 with derivative H^{cs} .*

Proof. We will first show that if $y \in \mathcal{W}_{loc}^s(x)$ and $h_{xy}^{cs} : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y)$ is differentiable at x then $Dh_{xy}^{cs}(x) = H_{xy}^{cs}$. We will prove this by contradiction.

If $D_x(h_{xy}^{cs}) \neq H_{xy}^{cs}$ then $L_{xy} := \Phi_y^{-1} \circ h_{xy}^{cs} \circ \Phi_x$ is differentiable at $0 \in E_x^u$ but $D_0(L_{xy}) \neq H_{xy}^{cs}$. Thus there exists $v \in E_x^u$ with $\|v\| = 1$ and some constants $\varepsilon_0, \eta > 0$ such that

$$\|L_{xy}(tv) - H_{xy}^s(tv)\| \geq \varepsilon_0 t, \quad \forall |t| \leq \eta.$$

By the uniform u -quasiconformality of f there is then a constant $C \geq 1$ independent of n such that

$$(4.8) \quad \|Df_y^n(L_{xy}(tv)) - Df_y^n(H_{xy}^s(tv))\| \geq C^{-1} \det(Df_y^n|_{E^u}) \varepsilon_0 t, \quad \forall |t| \leq \eta,$$

and also with the properties that for every $x \in M$ and any unit vector $\xi \in E_x^u$,

$$C^{-1} \det(Df_x^n|_{E^u}) \leq \|Df_x^n(\xi)\| \leq C \det(Df_x^n|_{E^u}),$$

and lastly the distortion estimate $\det(Df_y^n|_{E^u}) \leq C \det(Df_x^n|_{E^u})$ holds for $y \in \mathcal{W}_{loc}^s(x)$ and $n \geq 0$.

By the uniform continuity of the charts Φ_x in the x -variable, the \mathcal{W}^{cs} foliation and H^s , given any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for any $z \in M$, $w \in \mathcal{W}_{loc}^s(z)$ with $d_s(z, w) \leq \delta$ and any $\xi \in E_z^u$ satisfying $\|\xi\| \leq 1$ we have

$$(4.9) \quad \|L_{zw}(\xi) - H_{zw}^s(\xi)\| < \varepsilon$$

We choose $\varepsilon < C^{-3}\varepsilon_0$ and then choose n large enough that

$$d_s(f^n(x), f^n(y)) \leq \delta(\varepsilon),$$

and such that $C^{-1} \det(Df_x^n|_{E^u})^{-1} < \eta$.

We put $z = f^n(x)$ and $w = f^n(y)$. Applying the equivariance of Df with respect to the charts Φ_x , the center stable holonomy h^{cs} , and the linear stable holonomy H^s in equation (4.8) we obtain

$$\|L_{zw}(Df_x^n(tv)) - H_{zw}^s(Df_x^n(tv))\| \geq C^{-2} \det(Df_x^n|_{E^u}) \varepsilon_0 t, \quad \forall |t| \leq \eta,$$

Let t be the maximal number such that $\|Df_x^n(tv)\| \leq 1$ and then put $\xi = Df_x^n(tv)$. We conclude that equation (4.9) applies to the above and thus obtain

$$\varepsilon > C^{-2} \det(Df_x^n|_{E^u}) \varepsilon_0 t.$$

But we have $\eta \geq t \geq C^{-1} \det(Df_x^n|_{E^u})^{-1}$ by the uniform u -quasiconformality of f . Hence we conclude that $\varepsilon > C^{-3}\varepsilon_0$, contradicting our choice of ε .

Now suppose that $y \in \mathcal{W}_{loc}^{cs}(x)$ and h_{xy}^{cs} is differentiable at x . Let $z = \mathcal{W}_{loc}^{cu}(x) \cap \mathcal{W}_{loc}^s(y)$. Then $h_{xy}^{cs} = h_{zy}^{cs} \circ h_{xz}^{cs}$. The map $h_{xz}^{cs} : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(z)$ coincides with the center holonomy from $\mathcal{W}_{loc}^u(x)$ to $\mathcal{W}_{loc}^u(z)$ and thus it follows from Lemma 23 that h_{xz}^{cs} is a C^1 diffeomorphism with derivative H_{xz}^c at x . We conclude that h_{zy}^{cs} is differentiable at z and thus by our work above the derivative of h_{zy}^{cs} at z is given by H_{zy}^s . Thus $D_x(h_{xy}^{cs}) = H_{zy}^s \circ H_{xz}^c = H_{xy}^{cs}$. Since h_{xy}^{cs} is ACL from the quasiconformality given by Lemma 10 and H_{xy}^{cs} is uniformly continuous in x and y by the remarks preceding this lemma we conclude by Lemma 22 that h_{xy}^{cs} is C^1 with derivative given by H^{cs} . \square

LEMMA 25. *There is a continuous invariant conformal structure $\tau : M \rightarrow E^u$ for $Df|_{E^u}$ which is invariant under H^c , H^u , and H^s holonomies.*

Proof. By Proposition 18 there is a bounded measurable invariant conformal structure $\hat{\tau} : M \rightarrow E^u$ for $Df|_{E^u}$ defined on a full measure subset Ω of M such that $\hat{\tau}$ is H^u and H^s invariant on Ω . For a point $x \in M$ we define τ_x to be the barycenter in \mathcal{CE}_x^u in the nonpositively curved metric ρ_x of the set

$$\mathcal{O}_x = \left\{ (H_{xy}^c)^* \hat{\tau}_y : y \in \Omega \cap \mathcal{W}^c(x) \right\}$$

(for existence and uniqueness of barycenters of nonpositively curved metrics see [11]). The definition of τ_x assumes the set \mathcal{O}_x is nonempty; this will be true for m -a.e. $x \in M$ because of the absolute continuity of the center foliation. The definition of τ_x also assumes that \mathcal{O}_x is a *bounded* subset of \mathcal{CE}_x^u . This is clear under assumption (A) that the center foliation is uniformly compact.

For the case of assumption (B) when the center foliation has mostly noncompact leaves which are fixed by f we must give a different argument for the boundedness of \mathcal{O}_x . We define

$$\mathcal{O}_x^b = \left\{ (H_{xy}^c)^* \hat{\tau}_y : y \in \Omega \cap U_x \right\}$$

where we recall that U_x is the neighborhood of x in $\mathcal{W}^c(x)$ used to define the bundle \mathcal{E} . There is a uniform bound on the ρ_x -diameter of the sets \mathcal{O}_x^b from the reference conformal structure τ_x^0 on E_x^u . We write

$$\mathcal{O}_x = \bigcup_{n \in \mathbb{Z}} (Df_x^n)^* (\mathcal{O}_{f^n(x)}^b)$$

Since f is uniformly quasiconformal there is a uniform bound (independent of $n \in \mathbb{Z}$) on the distance from $(Df^n)^*(\mathcal{O}_{f^n(x)}^b)$ to τ_x^0 . It follows that the set \mathcal{O}_x is bounded in \mathcal{CE}_x^u .

Since

- H^c is equivariant with respect to Df according to equation (4.6),
- $\hat{\tau}$ is Df -invariant,
- $f(\mathcal{W}^c(x)) = \mathcal{W}^c(f(x))$ for every $x \in M$, and lastly
- Df^* is a fiberwise isometry on the conformal structure bundle \mathcal{CE}^u ,

we have $Df^*(\mathcal{O}_{f(x)}) = \mathcal{O}_x$ and consequently $Df^*\tau_{f(x)} = \tau_x$. Thus τ is also an invariant conformal structure. Clearly τ is invariant under H^c holonomy because of the composition property $H_{yz}^c \circ H_{xy}^c = H_{xz}^c$ for x, y, z in the same center leaf. By the equivariance of H^c with respect to H^u and H^s given by equations (4.1) and (4.2), τ is also invariant under H^s and H^u holonomy on Ω . In particular τ is invariant under both H^{cs} and H^u so τ is invariant under uniformly continuous holonomies along two transverse foliations of M . It follows that τ is uniformly continuous on Ω and thus has a unique continuous extension to M which is invariant under H^c , H^u , and H^s holonomies. \square

By combining Lemmas 24 and 25 we derive the main result of this subsection,

COROLLARY 26. *The center-stable holonomy h^{cs} between local unstable leaves is analytic in the charts $\{\Phi_x\}_{x \in M}$. Hence $h_{xy}^{cs} : \mathcal{W}_{loc}^u(x) \rightarrow \mathcal{W}_{loc}^u(y)$ is a C^∞ diffeomorphism*

Proof. Let τ be the continuous invariant conformal structure on E^u from Lemma 25 which is invariant under H^c , H^u and H^s . Let $y \in \mathcal{W}_{loc}^u(x)$. We consider τ_x and τ_y as conformal structures ω_x and ω_y on the Euclidean spaces E_x^u and E_y^u respectively by using the canonical identification for each $v \in E_x^u$ of $T_v E_x^u$ with E_x^u and assigning ω_x to be the image of τ_x in $T_v E_x^u$ under this identification.

We claim that $(\Phi_y^{-1} \circ h_{xy}^{cs} \circ \Phi_x)^* \omega_y = \omega_x$. To show this, let $v \in E_x^u$ be given and let $v' = \Phi_y^{-1}(h_{xy}^{cs}(\Phi_x(v))) \in E_y^u$ be its image. Let $z = \Phi_x(v)$ and $w = \Phi_y(v')$. Similarly to Lemma 23 we write

$$\begin{aligned} D_v(\Phi_y^{-1} \circ h_{xy}^{cs} \circ \Phi_x) &= D_0(\Phi_y^{-1} \circ \Phi_w) \\ &\circ D_0(\Phi_w^{-1} \circ h_{zw}^{cs} \circ \Phi_z) \circ D_v(\Phi_z^{-1} \circ \Phi_x) \\ &= H_{wy}^u \circ D_0(\Phi_w^{-1} \circ h_{zw}^{cs} \circ \Phi_z) \circ H_{xz}^u \\ &= H_{wy}^u \circ H_{zw}^{cs} \circ H_{xz}^u \end{aligned}$$

where in the third line we used the fact that H_{zw}^{cs} is the derivative of h_{zw}^{cs} at z from Lemma 24 and that both $D_0 \Phi_z = Id_{E_z^u}$ and $D_0 \Phi_w = Id_{E_w^u}$. By the invariance of τ under H^{cs} and H^u we conclude that $D_v(\Phi_y^{-1} \circ h_{xy}^{cs} \circ \Phi_x)^* \omega_y = \omega_x$ for every $v \in E_x^u$.

Identifying E_x^u and E_y^u with the Euclidean space \mathbb{R}^k , the inner products ω_x and ω_y are smoothly equivalent to the Euclidean norm on \mathbb{R}^k . Thus conformal mappings with respect to these inner products are the same as conformal mappings with respect to the standard Euclidean metric. Since $\Phi_y^{-1} \circ h_{xy}^{cs} \circ \Phi_x$ is conformal as a map between two open subsets of \mathbb{R}^k we conclude that it is analytic: for $k = 2$ this is a classical result in one-variable complex analysis and for $k \geq 3$ this follows from Gehring's theorem that all 1-quasiconformal mappings between subdomains of \mathbb{R}^k are the restrictions of Möbius transformations to these domains [15]. \square

4.4. Regularity of the foliations. We now prove higher regularity of the \mathcal{W}^{cu} , \mathcal{W}^{cs} , and \mathcal{W}^c foliations under additional bunching hypotheses on f . We begin with a folklore lemma which enables us to deduce regularity properties of a foliation from regularity properties of its holonomy maps between a specific family of transversals. When this family of transversals is smooth Lemma 27 follows directly from the claims in [27]; the proof of Lemma 27 is essentially identical to the proof of [29, Lemma 3.2] which handled the specific case of weak foliations of Anosov flows which were transverse to the strong unstable foliation.

LEMMA 27. *Let an integer $r \geq 1$ and $\alpha > 0$ be given. Suppose that \mathcal{W} and \mathcal{F} are two transverse foliations of M such that both \mathcal{W} and \mathcal{F} have uniformly $C^{r+\alpha}$ leaves. We further suppose that the local holonomy maps along \mathcal{W} between any two \mathcal{F} -leaves are locally uniformly $C^{r+\alpha}$. Then \mathcal{W} is a $C^{r+\alpha}$ foliation of M .*

Proof. Let $n = \dim M$ and $k = \dim \mathcal{W}$. As in [29, Lemma 3.2], we fix a point $x \in M$ together with a neighborhood V of x and choose a C^∞ coordinate chart $g : V \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $g(V \cap \mathcal{W}(x)) \subset \mathbb{R}^k \times \{0\}$ and $g(V \cap \mathcal{F}(x)) \subset \{0\} \times \mathbb{R}^{n-k}$. We then define for $p = (y, z) \in g(V)$,

$$\Psi(p) = (y, g(\mathcal{W})(p) \cap g(\mathcal{F})(0)) = (y, h_{p,0}(y)),$$

where $g(\mathcal{W})$, $g(\mathcal{F})$ denote the images of our foliations under g and $h_{p,0}(y)$ is the unique intersection point of $g(\mathcal{W})(p)$ with $g(\mathcal{F})(0)$ inside of $g(V)$. This map straightens the \mathcal{W} -foliation into a foliation of $\mathbb{R}^k \times \mathbb{R}^{n-k}$ by k -disks $D^k \times \{z\}$. Since the leaves of \mathcal{W} are uniformly $C^{r+\alpha}$ the map Ψ is $C^{r+\alpha}$ when restricted to the leaves of \mathcal{W} , and since the holonomy maps of the \mathcal{W} foliation between \mathcal{F} -transversals are uniformly $C^{r+\alpha}$ the chart Ψ is also $C^{r+\alpha}$ along the leaves of \mathcal{F} . By Journé's lemma [20] this implies that Ψ is $C^{r+\alpha}$. \square

LEMMA 28. *Suppose f is r -bunched for some $r \geq 1$, then there is an $\alpha > 0$ such that \mathcal{W}^c , \mathcal{W}^{cs} , and \mathcal{W}^{cu} are $C^{r+\alpha}$ foliations of M . If f is ∞ -bunched then these foliations are all C^∞ .*

Proof. Since f is C^∞ and r -bunched, there is an $\alpha > 0$ such that the leaves of \mathcal{W}_{loc}^* , $*$ $\in \{cs, cu, c\}$ are uniformly $C^{r+\alpha}$. By Corollary 26 the cs -holonomy maps between local unstable leaves are analytic diffeomorphisms. Hence using \mathcal{W}^u as our transverse foliation \mathcal{F} for Lemma 27 we conclude that \mathcal{W}^{cs} is a $C^{r+\alpha}$ foliation of M .

By applying all of the results of this section to the cs -holonomy maps of f^{-1} instead (i.e., the cu -holonomy maps of f) we conclude that the cu -holonomy maps are analytic between local stable leaves. Hence we also obtain that \mathcal{W}^{cu} is a $C^{r+\alpha}$ foliation of M .

For each $x \in M$ and $m = \dim E^u$, $k = \dim E^c$, $n = \dim M$ we can thus find a neighborhood V of x and a $C^{r+\alpha}$ foliation chart

$$\Psi : V \rightarrow D^{m+k} \times D^{n-m-k} = D^m \times D^{n-m} \subset \mathbb{R}^n,$$

such that \mathcal{W}^{cu} is mapped to the foliation by $(m+k)$ -cubes $D^{m+k} \times \{z\}$, $z \in D^{n-m-k}$ and \mathcal{W}^{cs} is mapped to the foliation by $(n-m)$ -cubes $\{y\} \times D^{n-m}$, $y \in D^m$ (here D^j again denotes the open unit cube in \mathbb{R}^j). The intersection of these two foliations is the image of \mathcal{W}^c which is a foliation by k -disks $\{y'\} \times D^k \times \{z'\}$, $y' \in D^m$,

$z' \in D^{n-m-k}$. Thus Ψ is also a $C^{r+\alpha}$ foliation chart for E^c and therefore \mathcal{W}^c is also a $C^{r+\alpha}$ foliation of M . \square

Our final lemma applies under both assumptions (A) and (B) in the case that the center is 1-dimensional. and is a straightforward consequence of the C^1 regularity of the center foliation together with the fact that in dimension 1, length and volume are the same.

LEMMA 29. *If $\dim E^c = 1$ then f is ∞ -bunched and therefore \mathcal{W}^c , \mathcal{W}^{cs} , and \mathcal{W}^{cu} are C^∞ foliations of M . Furthermore there is a C^∞ norm $|\cdot|$ on E^c with respect to which $Df|_{E^c}$ acts by isometries.*

Proof. When $\dim E^c = 1$, f is always 1-bunched. Hence by Lemma 28 the center foliation \mathcal{W}^c is $C^{1+\alpha}$ for some $\alpha > 0$. Let ν_x^c denote the Riemannian volume on $U_x \subset \mathcal{W}^c(x)$. Since \mathcal{W}^c is a $C^{1+\alpha}$ foliation the conditional measures $\{m_x^c\}_{x \in M}$ of the volume m on the sets U_x are absolutely continuous with continuous densities with respect to ν_x^c . Thus there are positive continuous functions $\zeta_x : U_x \rightarrow \mathbb{R}$ such that $dm_x^c = \zeta_x d\nu_x^c$ which also depend continuously on $x \in M$.

Since $f_* m_x^c = m_{f(x)}^c$ and $f_* \nu_x^c(y) = \|Df_y|_{E_y^c}\|^{-1} \nu_{f(x)}^c(y)$ we thus derive the relationship

$$\frac{\zeta_x(y)}{\zeta_{f(x)}(f(y))} = \|Df_y|_{E_y^c}\|,$$

which is valid for $y \in U_x$.

We set $\zeta(x) := \zeta_x(x)$. Thus $\zeta : M \rightarrow (0, \infty)$ is a measurable function satisfying the equation,

$$\frac{\zeta(x)}{\zeta(f(x))} = \|Df_x|_{E_x^c}\|,$$

for every $x \in M$. It is then clear that $Df|_{E^c}$ acts by isometries with respect to the norm $|\cdot| = \zeta \cdot \|\cdot\|$ on E^c .

We will first show that ζ is continuous on M . By [27] the center bundle E^c is β -Hölder continuous for some $\beta > 0$. Thus the 1-dimensional linear cocycle $Df|_{E^c}$ admits continuous stable and unstable holonomies P^s and P^u according to Proposition 7. Let $\{K_n\}_{n \geq 1}$ be an increasing sequence of subsets of M such that ζ is uniformly continuous on K_n and $m(M \setminus K_n) < 2^{-n}$. By Lemma 17 there is a full measure subset Ω of M such that if $x, y \in \Omega$ then there is an $n > 0$ and there are infinite sequences $n_k \rightarrow \infty$ and $n'_k \rightarrow \infty$ such that $f^{n_k}(x), f^{n_k}(y) \in K_n$ and $f^{-n'_k}(x), f^{-n'_k}(y) \in K_n$.

Then for $x, y \in \Omega$ with $y \in \mathcal{W}_{loc}^s(x)$,

$$\frac{\zeta(x)}{\zeta(y)} = \frac{\zeta(f^{n_k}(x))}{\zeta(f^{n_k}(y))} \cdot \frac{\|Df_x^{n_k}|_{E_x^c}\|}{\|Df_y^{n_k}|_{E_y^c}\|}.$$

Letting $n_k \rightarrow \infty$ we have $\frac{\zeta(f^{n_k}(x))}{\zeta(f^{n_k}(y))} \rightarrow 1$ and $\frac{\|Df_x^{n_k}|_{E_x^c}\|}{\|Df_y^{n_k}|_{E_y^c}\|} \rightarrow P_{xy}^s$. Hence we conclude that

$$\zeta(x) = P_{xy}^s \zeta(y).$$

Similarly if $y \in \mathcal{W}_{loc}^u(x)$ we conclude instead that $\zeta(x) = P_{xy}^u \zeta(y)$. The holonomies P^s and P^u are uniformly continuous on local stable and unstable leaves respectively so this shows that ζ restricted to Ω is uniformly continuous on local stable and unstable leaves.

For $y \in \mathcal{W}_{loc}^c(x)$ we recall from the construction of the measures m_x^c in Proposition 16 that there is a unique positive constant P_{xy}^c such that $P_{xy}^c \cdot m_x^c$ and m_y^c coincide on the overlap $U_x \cap U_y$ of the two neighborhoods on which these measures are each supported. Because the neighborhoods $\{U_y\}_{y \in \mathcal{W}^c(x)}$ vary continuously over points in the same center leaf and these are the neighborhoods used to normalize the measures m_y^c in Proposition 16 we obtain that P_{xy}^c varies uniformly continuously in x and y for $y \in \mathcal{W}_{loc}^c(x)$.

The measures ν_x^c and ν_y^c clearly also coincide on the overlap $U_x \cap U_y$ so it follows that $P_{xy}^c \zeta_x = \zeta_y$ on $U_x \cap U_y$. Hence for $y \in \mathcal{W}_{loc}^c(x)$ we have $\zeta(y) = P_{xy}^c \zeta_x(y)$. Thus ζ is also uniformly continuous when restricted to the center foliation. Since ζ is uniformly continuous when restricted to each of the three foliations \mathcal{W}^s , \mathcal{W}^u , and \mathcal{W}^c we conclude ζ is continuous on M .

Hence there is a continuous norm on $|\cdot|$ on E^c with respect to which $Df|_{E^c}$ acts by isometries. This implies that f is r -bunched for every $r \geq 1$, i.e., f is ∞ -bunched. Thus by Lemma 28 \mathcal{W}^{cu} , \mathcal{W}^{cs} , and \mathcal{W}^c are C^∞ foliations of M . This implies that the conditional measures $\{\hat{m}_x^c\}_{x \in M}$ of m from Proposition 16 on the \mathcal{W}^c are both C^∞ in the basepoint $x \in M$ and are C^∞ equivalent to the smooth Riemannian arclength ν_x^c on $\mathcal{W}^c(x)$; for this assertion recall that we assume that m is smoothly equivalent to the Riemannian volume on M .

In the case of assumption (A) this implies without further argument that the family of conditional measures $\{m_x^c\}_{x \in M}$ used in this proof are also C^∞ in $x \in M$ and are C^∞ equivalent to ν_x^c , since there is a canonical smooth normalization of the family $\{\hat{m}_x^c\}_{x \in M}$ such that $m_x^c(\mathcal{W}^c(x)) = 1$ for each $x \in M$. In the case of assumption (B) we only need to note that the arcs $U_x \subset \mathcal{W}^c(x)$ are determined by their endpoints in a canonical smooth fashion according to the discussion at the beginning of Section 4 and these endpoints are given by $f^{-1}(x)$ and $f(x)$, which clearly smoothly depend on x . Hence there is a smooth normalization of the family $\{\hat{m}_x^c\}_{x \in M}$ of conditional measures such that $m_x^c(U_x) = 1$ for every $x \in M$ and we obtain the same conclusion as we did in the case of assumption (A). As a consequence the family of C^∞ functions $\zeta_x : U_x \rightarrow (0, \infty)$ is also C^∞ in the basepoint x , so we conclude that $\zeta(x) = \zeta_x(x)$ is C^∞ and consequently the norm $|\cdot|$ on E^c is C^∞ . \square

4.5. Proofs of Theorems 2-4.

Proof of Theorem 2. Since f is dynamically coherent, r -bunched, volume-preserving, and uniformly quasiconformal with uniformly compact center foliation, we conclude from the results of Section 4 that the foliations \mathcal{W}^{cs} , \mathcal{W}^{cu} and \mathcal{W}^c are $C^{r+\alpha}$ for some $\alpha > 0$. We define N to be the quotient of M by the equivalence relation $x \equiv y$ if $y \in \mathcal{W}^c(x)$. Since the center foliation is uniformly compact, N is a topological manifold, and since \mathcal{W}^c is a C^r foliation of M we actually conclude that N is a C^r manifold and f descends to a $C^{r+\alpha}$ Anosov diffeomorphism $g : N \rightarrow N$.

The invariance of the conformal structure τ from Lemma 25 under center holonomy implies that τ descends to a conformal structure $\bar{\tau}$ on the unstable bundle

of g acting on N . This shows that g is uniformly u -quasiconformal. An analogous argument using the invariant conformal structure on E^s shows that g is also uniformly s -quasiconformal. Hence g is a $C^{r+\alpha}$ uniformly quasiconformal Anosov diffeomorphism of N . Thus g satisfies Hasselblatt's pinching conditions for $C^{1+\alpha}$ regularity of the Anosov splitting [18] and we conclude that the stable and unstable foliations $\mathcal{W}^{s,g}$ and $\mathcal{W}^{u,g}$ of g are $C^{1+\alpha}$.

Our remaining task is to show that the stable and unstable foliations \mathcal{W}^s and \mathcal{W}^u of f are $C^{1+\alpha}$. By Lemma 27 it suffices to show that the local stable holonomy maps of \mathcal{W}^s between leaves of the center-unstable foliation \mathcal{W}^{cu} are $C^{1+\alpha}$. The proof of this same fact for the local unstable holonomy maps will be analogous. Since f is r -bunched for some $r \geq 1$, by [27] the stable holonomy maps between \mathcal{W}^c leaves are $C^{1+\alpha}$ diffeomorphisms for some (possibly smaller) $\alpha > 0$. Since \mathcal{W}^s and \mathcal{W}^u project under the smooth submersion $\pi : M \rightarrow N$ to $C^{1+\alpha}$ foliations $\mathcal{W}^{s,g}$ and $\mathcal{W}^{u,g}$ of N we conclude that the local stable holonomy maps $\mathcal{W}_{loc}^{cu}(x) \rightarrow \mathcal{W}_{loc}^{cu}(y)$ are also $C^{1+\alpha}$ when restricted to the subfoliation \mathcal{W}^u of \mathcal{W}^{cu} . Hence the local stable holonomy maps between center-unstable leaves are $C^{1+\alpha}$ when restricted to the two subfoliations \mathcal{W}^c and \mathcal{W}^u of \mathcal{W}^{cu} so it follows by Journé's lemma that these local stable holonomy maps themselves are $C^{1+\alpha}$ and thus \mathcal{W}^s is a $C^{1+\alpha}$ foliation of M . This completes the proof of claims (1) and (2) of Theorem 2.

We now assume that f is ∞ -bunched. Applying the results of the previous two paragraphs with $r = \infty$ we conclude that \mathcal{W}^{cs} , \mathcal{W}^{cu} and \mathcal{W}^c are C^∞ foliations of M , N is a C^∞ manifold, and $g : N \rightarrow N$ is a C^∞ volume-preserving uniformly quasiconformal Anosov diffeomorphism. By the classification theorem of Fang [12] g is smoothly conjugate to a hyperbolic toral automorphism and the stable and unstable foliations $\mathcal{W}^{s,g}$ and $\mathcal{W}^{u,g}$ of g are C^∞ . Lastly, to show that the stable and unstable foliations \mathcal{W}^s and \mathcal{W}^u of f are C^∞ we repeat the argument of the previous paragraph with the $C^{1+\alpha}$ -regularity replaced with C^∞ -regularity for the stable and unstable foliations of g . \square

Proof of Corollary 3. Since f is uniformly quasiconformal with uniformly compact center foliation and $\dim E^c = 1$, by Lemma 29 f is ∞ -bunched and there is a smooth norm $|\cdot|$ on E^c such that $Df|_{E^c}$ is an isometry with respect to this norm. Hence the conclusions of part (3) of Theorem 2 apply to f so that the foliations \mathcal{W}^c , \mathcal{W}^u , and \mathcal{W}^s of M are C^∞ , the quotient $\pi : M \rightarrow N$ of M by the center foliation is a torus and there is a hyperbolic toral automorphism $g : N \rightarrow N$ such that $\pi \circ f = g \circ \pi$. Since there is a smooth norm on E^c with respect to which $Df|_{E^c}$ acts by isometries we conclude that f is an isometric extension of g . \square

Proof of Theorem 4. We will first prove that the foliations \mathcal{W}^{cs} , \mathcal{W}^{cu} , and \mathcal{W}^c are C^∞ for any C^1 -small enough volume-preserving uniformly quasiconformal perturbation f of ψ_1 under the assumption that ψ_t itself has no periodic orbits of period ≤ 2 .

We claim that there is a C^1 -open neighborhood \mathcal{U} of ψ_1 in the space of smooth volume-preserving diffeomorphisms of M such that if $f \in \mathcal{U}$ then f satisfies assumption (B) of Section 4. Since ψ_1 is partially hyperbolic, the center foliation \mathcal{W}^{c,ψ_1} for ψ_1 is normally hyperbolic and every center leaf is fixed by ψ_1 , by the work of Hirsch, Pugh, Shub [19] we deduce that any f which is C^1 close to ψ_1 is partially hyperbolic, dynamically coherent, and has the property that $f(\mathcal{W}^c(x)) = \mathcal{W}^c(x)$ for every $x \in M$. Furthermore the center bundle E^c for f is

orientable with orientation preserved by f because it is C^0 close to the orientable center bundle for ψ_1 . Since ψ_t has no periodic orbits of period ≤ 1 , ψ_1 has no fixed points and thus if the neighborhood \mathcal{U} is chosen small enough then f will have no fixed points as well. Finally, consider for each $x \in M$ the flow line U_x^ψ of x in the center foliation \mathcal{W}^{c,ψ_1} of ψ_1 given by $U_x^\psi = \bigcup_{t \in [-1,1]} \psi_t(x)$. Since ψ_t has no periodic orbits of period ≤ 2 , U_x^ψ is a subarc of $\mathcal{W}^{c,\psi_1}(x)$ which is not a circle. The subarc of U_x of the center leaf $\mathcal{W}^c(x)$ for f through x constructed at the beginning of Section 4 is uniformly close to U_x^ψ and thus if f is C^1 close enough to ψ_1 then U_x will be a subarc of $\mathcal{W}^c(x)$ instead of a circle. Thus f does not wrap and so f satisfies assumption (B) of Section 4.

By Proposition 29 there is thus a C^∞ norm $|\cdot|$ on E^c with respect to which $Df|_{E^c}$ is an isometry and we also conclude that the invariant foliations \mathcal{W}^{cs} , \mathcal{W}^{cu} , and \mathcal{W}^c for f are C^∞ . Since E^c is a smooth orientable subbundle of TM we can find a smooth nonvanishing section $Z : M \rightarrow E^c$.

Now suppose only that there is a finite cover $p : \hat{M} \rightarrow M$ of M such that the lift $\hat{\psi}_t$ of ψ_t to \hat{M} has no periodic orbits of period ≤ 2 . Let $\hat{\mathcal{U}}$ be the C^1 -open neighborhood of $\hat{\psi}_1$ given by Theorem 4 applied to $\hat{\psi}_t$. We let \mathcal{U} be the C^1 -open neighborhood of ψ_1 consisting of all smooth volume-preserving diffeomorphisms f whose lift $\hat{f} : \hat{M} \rightarrow \hat{M}$ lies in $\hat{\mathcal{U}}$. Then the lifts $\hat{\mathcal{W}}^{cs}$, $\hat{\mathcal{W}}^{cu}$, and $\hat{\mathcal{W}}^c$ of the invariant foliations for f are C^∞ foliations of \hat{m} and thus since the projection p is a local smooth diffeomorphism we conclude that the invariant foliations \mathcal{W}^{cs} , \mathcal{W}^{cu} , and \mathcal{W}^c for f are C^∞ .

We take Z to be a nonvanishing C^∞ section of E^c . Since f preserves the orientation of E^c and fixes every center leaf we can, by replacing Z with $-Z$ if necessary, arrange that for every $x \in M$, $Z(x)$ is oriented in the direction of $f(x)$ on $\mathcal{W}^c(x)$. Let φ_t^Z be the C^∞ flow generated by Z . Its flowlines are the center foliation \mathcal{W}^c . There is then a positive smooth function $\sigma : M \rightarrow (0, \infty)$ such that $\varphi_{\sigma(x)}^Z = f(x)$ for every $x \in M$. We then set $\varphi_t(x) = \varphi_{t, \sigma(x)}^Z$. Then φ_t is a C^∞ flow on M with $\varphi_1 = f$. The fact that φ_t is a uniformly quasiconformal Anosov flow follows from the partial hyperbolicity and uniform quasiconformality estimates for its time-1 map f .

Finally we show that φ_t preserves a measure smoothly equivalent to volume. For each $t \in \mathbb{R}$, $\varphi_t : M \rightarrow M$ is a smooth diffeomorphism and thus the measures $(\varphi_t)_* m$ and m are smoothly equivalent with C^∞ Radon-Nikodym derivative $\frac{d(\varphi_t)_* m}{dm} := J_t$. Since $\varphi_1 = f$ we have $J_1 \equiv 1$. For every $x \in M$ we clearly have $J_{t+s}(x) = J_t(\varphi_s(x)) \cdot J_s(x)$ from the property that $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

We claim that φ_t is topologically transitive. Since $\varphi_1 = f$ is volume-preserving the nonwandering set of φ_t is all of M . By the spectral decomposition theorem for flows [31] we can decompose M into connected components invariant under φ_t on which φ_t is topologically transitive; since M is connected we conclude that φ_t is actually topologically transitive on M . We can thus apply the following criterion for a topologically transitive Anosov flow φ_t to preserve a measure smoothly equivalent to volume: φ_t preserves a measure smoothly equivalent to volume if and only if for every periodic point p of φ_t of period $\ell(p)$ we have $J_{\ell(p)}(p) = 1$ [26].

Suppose that this does not hold, then without loss of generality we can assume that there is a periodic point p for which $J_{\ell(p)}(p)$. For this point and any integer $n > 0$ we then have

$$J_{n\ell(p)}(p) = (J_{\ell(p)}(p))^n \rightarrow \infty,$$

as $n \rightarrow \infty$. On the other hand, let $\lfloor n\ell(p) \rfloor$ denote the greatest integer smaller than $n\ell(p)$ and let $K := \sup_{0 \leq t \leq 1} \sup_{x \in M} J_t(x)$. Then since $J_1 \equiv 1$ we have

$$J_{n\ell(p)}(p) = J_{n\ell(p) - \lfloor n\ell(p) \rfloor}(p) \leq K < \infty,$$

for each integer $n > 0$. Thus we obtain a contradiction so that $J_{\ell(p)}(p) = 1$ for every periodic point p and thus φ_t preserves a measure smoothly equivalent to volume on M . \square

5. PROOF OF THEOREM 1

Let X be a closed Riemannian manifold of constant negative curvature with $\dim X \geq 3$ and let T^1X be the unit tangent bundle of X . We let $\pi : T^1X \rightarrow X$ denote the standard projection of a unit tangent vector to its basepoint in X . We let ψ_t denote the geodesic flow on T^1X and consider a smooth, volume-preserving perturbation f of the time-1 map ψ_1 . We will establish in this section that the equalities $\lambda_+^u = \lambda_-^u$ and $\lambda_+^s = \lambda_-^s$ imply that $Df|_{E^u}$ and $Df|_{E^s}$ respectively are uniformly quasiconformal for small enough volume-preserving perturbations of ψ_1 . We will prove this implication for the unstable bundle E^u ; the proof for E^s will be analogous. By Theorem 4 and the smooth orbit equivalence classification result of Fang [14] this suffices to complete the proof of Theorem 1 from the Introduction.

We first need to recall some properties of the frame flow associated to closed Riemannian manifolds of constant negative curvature. Let $X^{(2)}$ be the 2-frame bundle over X which has fiber over each $p \in X$ given by

$$X_p^{(2)} = \{(v, w) \in T_p^1X : v \text{ is orthogonal to } w\}.$$

We let $\psi_t^{(2)}$ be the 2-frame flow on $X^{(2)}$ obtained by applying the geodesic flow ψ_t to the first vector $v \in T_p^1X$ and then taking the image of w under parallel transport along the geodesic $\gamma(s) = \pi(\psi_s(v))$, $s \in [0, t]$, on X .

We let $E^{u,\psi}$ be the unstable bundle of the geodesic flow ψ_t on T^1X and we let $SE^{u,\psi}$ be the unit sphere inside of $E^{u,\psi}$, where we equip $E^{u,\psi}$ with the Riemannian norm $\|\cdot\|$ coming from its realization as the tangent spaces of unstable horospheres in the universal cover of X . We have a smooth identification $SE^{u,\psi} \rightarrow X^{(2)}$ coming from this realization by identifying a unit vector $v \in T_p^1X$ together with a unit vector $w \in SE_v^{u,\psi}$ to the orthonormal 2-frame $(v, w) \in X_p^{(2)}$ obtained from identifying w with its image in the tangent space of the unstable horosphere through p which is orthogonal to v . Since X has constant negative curvature the geodesic flow is conformal on unstable horospheres and therefore under this identification the 2-frame flow $\psi_t^{(2)}$ corresponds to the renormalized derivative action $w \rightarrow \frac{D\psi_t(w)}{\|D\psi_t(w)\|}$ on $SE^{u,\psi}$. For a more detailed version of this discussion as well as the discussion in the paragraphs below we refer to [6], [8].

We consider the stable and unstable holonomies $H^{s,\psi}$ and $H^{u,\psi}$ of ψ_t on $E^{u,\psi}$ and their renormalized versions $SH^{s,\psi}(\cdot) = \frac{H^{s,\psi}(\cdot)}{\|H^{s,\psi}(\cdot)\|}$, $SH^{u,\psi}(\cdot) = \frac{H^{u,\psi}(\cdot)}{\|H^{u,\psi}(\cdot)\|}$ which give isometric identifications $SE_v^{u,\psi} \rightarrow SE_{v'}^{u,\psi}$ for $v' \in \mathcal{W}^{s,\psi}(v)$ and $v' \in \mathcal{W}^{u,\psi}(v)$ respectively, where $\mathcal{W}^{s,\psi}$ and $\mathcal{W}^{u,\psi}$ denote the stable and unstable foliations of ψ respectively.

An *su-loop* based at $v \in T^1X$ is an *su-path* for ψ_t which starts and ends at v . Based on the discussion of the previous paragraphs, given an *su-loop* γ for ψ_t based at a point $v \in T^1X$ we can associate an isometry $T_\psi(\gamma) : SE_v^{u,\psi} \rightarrow SE_v^{u,\psi}$ obtained by composing the renormalized stable and unstable holonomy maps $SH_{v_i v_{i+1}}^{s,\psi} : SE_{v_i}^{u,\psi} \rightarrow SE_{v_{i+1}}^{u,\psi}$ and $SH_{v_j v_{j+1}}^{s,\psi} : SE_{v_j}^{u,\psi} \rightarrow SE_{v_{j+1}}^{u,\psi}$ along this loop, where $\gamma_{v_i v_{i+1}} \subset \mathcal{W}^{s,\psi}(v_i)$ and $\gamma_{v_j v_{j+1}} \subset \mathcal{W}^{u,\psi}(v_j)$. Thus, identifying $SE_v^{u,\psi}$ with the unit sphere S^{n-1} in \mathbb{R}^n for $n := \dim X - 1$, $T_\psi(\gamma)$ gives us an element of the special orthogonal group $SO(n)$.

The key observation due to Brin and Karcher [6] is that for a closed constant negative curvature manifold X and any $v \in T^1X$ there are finitely many *su-loops* $\gamma_1, \dots, \gamma_k$ such that $T_\psi(\gamma_1), \dots, T_\psi(\gamma_k)$ generate $SO(n)$ as a Lie group when we identify E_v^u . Moreover the number k of loops used and the total lengths of these loops may both be taken to be bounded independently of the point v . As a consequence we have the following proposition,

PROPOSITION 30. *For any $\delta > 0$ there is a constant $L > 0$ and an integer $\ell > 0$ such that given any $v \in T^1X$ there is a finite collection $\gamma_1, \dots, \gamma_\ell$ of *su-loops* based at v of total length at most L for which the collection of points $\{T_\psi(\gamma_i)(w)\}_{i=1}^\ell$ is δ -dense in $SE_v^{u,\psi}$ for any $w \in SE_v^{u,\psi}$.*

Proof. Fix a $\frac{\delta}{2}$ -dense collection $\{w_j\}_{j=1}^k$ of points in $SE_v^{u,\psi}$. Since there are finitely many *su-loops* based at v whose associated isometries generate $SO(n)$ as a Lie group and $SO(n)$ acts transitively on SE_v^u , there is a finite collection $\gamma_1, \dots, \gamma_\ell$ of *su-loops* based at v for which each of the sets $\{T_\psi(\gamma_i)(w_j)\}_{i=1}^\ell$ for $1 \leq j \leq k$ is $\frac{\delta}{2}$ -dense in $SE_v^{u,\psi}$.

Now let w be any point in $SE_v^{u,\psi}$. Then there is some w_j such that $\|w - w_j\| < \frac{\delta}{2}$. Since each $T_\psi(\gamma_i)$ is an isometry we then also have $\|T_\psi(\gamma_i)(w_j) - T_\psi(\gamma_i)(w)\| < \frac{\delta}{2}$ for each $1 \leq i \leq \ell$. This implies that $\{T_\psi(\gamma_i)(w)\}_{i=1}^\ell$ is a δ -dense subset of $SE_v^{u,\psi}$. \square

Let f be a C^1 -small perturbation of the time-1 map ψ_1 . If this perturbation is small enough then the linear cocycle $Df|_{E^u}$ is *fiber bunched* and consequently the conclusions of Proposition 7 apply to $Df|_{E^u}$, see [22, Proposition 4.2]. Thus the linear cocycle $Df|_{E^u}$ admits linear stable and unstable holonomies H^s and H^u . For $v \in T^1X$ we define $\mathbb{P}E_v^u$ to be the projective space of E_v^u and we define $\mathbb{P}H^s$ and $\mathbb{P}H^u$ to be the induced maps of H^s and H^u on the projective spaces $\mathbb{P}E_v^u \rightarrow \mathbb{P}E_{v'}^u$ for $v' \in \mathcal{W}^s(v)$ and $v' \in \mathcal{W}^u(v)$ respectively, where now \mathcal{W}^s and \mathcal{W}^u denote the stable and unstable foliations of f . We let $\mathbb{P}Df_v : \mathbb{P}E_v^u \rightarrow \mathbb{P}E_{f(v)}^u$ be the induced map from Df_v .

We obtain below a version of Proposition 30 which also applies to the perturbation f provided that this perturbation is small enough. We endow $\mathbb{P}E_v^u$ with the

Riemannian metric induced from the Riemannian metric on E^u which is in turn induced from the metric on T^1X . Given an su -loop γ for f based at $v \in T^1X$ we associate the map $T(\gamma) : \mathbb{P}E_v^u \rightarrow \mathbb{P}E_v^u$ obtained by composing the projectivized stable and unstable holonomies $\mathbb{P}H^s$ and $\mathbb{P}H^u$ along the segments of this loop which lie in the stable and unstable leaves of f respectively. Unlike the case of T_ψ above, $T(\gamma)$ is not necessarily an isometry of $\mathbb{P}E_v^u$.

We will need the following proposition that follows from results of Katok and Kononenko,

PROPOSITION 31 ([24]). *Let $\psi_t : M \rightarrow M$ be a contact Anosov flow on a closed Riemannian manifold. Then there is a C^2 -open neighborhood \mathcal{V} of ψ_1 in the space of C^2 diffeomorphisms of M and an integer $J > 0$ such that for every $\varepsilon > 0$ and every $f \in \mathcal{V}$ there exists an $\eta > 0$ such that for every $p, q \in M$ with $d(p, q) < \eta$, there exists a J -legged su -path from p to q of length less than ε .*

Recall that for each pair of nearby points $x, y \in T^1X$ we let $I_{xy} : E_x^u \rightarrow E_y^u$ be a linear identification which is Hölder close to the identity. This induces an identification $\mathbb{P}I_{xy} : \mathbb{P}E_x^u \rightarrow \mathbb{P}E_y^u$ that is Hölder close to the identity in x and y .

LEMMA 32. *Given any $\delta > 0$ there is a C^2 -open neighborhood \mathcal{U} of ψ_1 such that if $f \in \mathcal{U}$ then for any $v \in T^1X$ there is a finite collection $\gamma_1, \dots, \gamma_\ell$ of su -loops for f based at v such that the collection of points $\{T(\gamma_i)(w)\}_{i=1}^\ell$ is δ -dense in $\mathbb{P}E_v^u$ for any $w \in \mathbb{P}E_v^u$.*

Proof. Let $\delta > 0$ be given. We first apply Proposition 30 to ψ_t to obtain a constant $L > 0$ and integer $\ell > 0$ such that for any $v \in T^1X$ there is a collection of su -loops $\sigma_1, \dots, \sigma_\ell$ based at v of total length at most L such that $\{T_\psi(\sigma_i)(w)\}_{i=1}^\ell$ is $\frac{\delta}{3}$ -dense in $SE_v^{u,\psi}$ for any $w \in SE_v^{u,\psi}$.

We apply Proposition 31 for a small $\varepsilon > 0$ to be determined. Given the $\eta > 0$ obtained from Proposition 31 for this ε we claim that we can find a C^2 -open neighborhood \mathcal{U}' of ψ_1 such that for each $v \in T^1X$ there are points v_1, \dots, v_ℓ satisfying $d(v, v_i) < \eta$ and for each $1 \leq i \leq \ell$ there is an su -path β_i for f from v to v_i such that the collection $\{\mathbb{P}I_{v_i v} \circ T(\beta_i)(w)\}_{i=1}^\ell$ is $\frac{2\delta}{3}$ -dense in $\mathbb{P}E_v^u$ for any $w \in \mathbb{P}E_v^u$. This follows from the facts that the stable and unstable foliations \mathcal{W}^s and \mathcal{W}^u depend continuously on f in the C^2 topology and the stable and unstable holonomies H^s and H^u of $Df|_{E^u}$ also depend continuously on f in the C^2 topology[2]. Hence we obtain this statement by considering su -paths $\beta_1, \dots, \beta_\ell$ for f which are close enough to the su -loops $\sigma_1, \dots, \sigma_\ell$ for ψ_t ; we can make these paths as close as desired to the loops for ψ_t by making the neighborhood \mathcal{U} small enough.

For each $1 \leq i \leq \ell$ we let γ_i be the su -loop based at v for f obtained by concatenating β_i with the J -legged su -path of length less than ε connecting v_i to v given by Proposition 31. Since the number of legs J is fixed and both H_{xy}^u and H_{xy}^s converge uniformly to the identity as y converges to x for $x, y \in T^1X$ we conclude that if ε is small enough (independent of the choice of $v \in T^1X$) then the collection of points $\{T(\gamma_i)(w)\}_{i=1}^\ell$ is δ -dense in $\mathbb{P}E_v^u$ for any $w \in \mathbb{P}E_v^u$. \square

The use of Lemma 32 is the reason that we lose C^1 -openness of the neighborhood \mathcal{U} in Theorem 1.

It is easy to see that there is a $\delta_0 > 0$ with the property that if V_1 and V_2 are any two proper linear subspaces of \mathbb{R}^n then the union $\mathbb{P}V_1 \cup \mathbb{P}V_2$ of their projectivizations in $\mathbb{R}\mathbb{P}^{n-1}$ is *not* δ_0 -dense. Thus it follows that there is a $\delta > 0$ and a C^1 -open

neighborhood \mathcal{U}' of ψ_1 such that for any $f \in \mathcal{U}'$, any $v \in T^1X$, and any pair of proper linear subspaces V_1 and V_2 in E_v^u , the union $\mathbb{P}V_1 \cup \mathbb{P}V_2$ is not δ -dense in $\mathbb{P}E_v^u$. We apply Lemma 32 with this δ and let $f \in \mathcal{U} \subset \mathcal{U}'$ be a smooth volume-preserving diffeomorphism in the resulting open neighborhood with the property that $\lambda_+^u = \lambda_-^u$.

We will now show that $Df|_{E^u}$ is uniformly quasiconformal to complete the proof of Theorem 1. Since ψ_1 is a stably accessible partially hyperbolic diffeomorphism (this well-known fact can be derived as a consequence of Proposition 31) we may assume that f is also an accessible partially hyperbolic diffeomorphism. Since the neighborhood \mathcal{U} is chosen small enough that $Df|_{E^u}$ satisfies the fiber bunching condition that guarantees the existence of the stable and unstable holonomies H^s and H^u we conclude by the work of Avila, Santamaria and Viana [2] that the equality $\lambda_+^u = \lambda_-^u$ implies that there is a $\mathbb{P}Df$ -invariant probability measure μ on $\mathbb{P}E^u$ projecting down to the invariant volume m for f on T^1X and which has a disintegration $\{\mu_v\}_{v \in T^1X}$ into probability measures μ_v on the projective fibers $\mathbb{P}E_v^u$ which depend continuously on the basepoint v . Furthermore this disintegration is invariant under the projected stable and unstable holonomy, that is to say, if $v' \in \mathcal{W}^s(v)$ then $(\mathbb{P}H_{vv'}^s)_*\mu_v = \mu_{v'}$ and a similar equation holds for $\mathbb{P}H^u$.

Suppose that $Df|_{E^u}$ is not uniformly quasiconformal. Then there is a point $v \in T^1X$, unit vectors $w_1, w_2 \in E_v^u$, and a sequence $n_k \rightarrow \infty$ such that $\frac{\|Df^{n_k}(w_1)\|}{\|Df^{n_k}(w_2)\|} \rightarrow \infty$ as $n_k \rightarrow \infty$. By passing to a further subsequence and using the compactness of T^1X we can assume that there is some $z \in T^1X$ such that $f^{n_k}(v) \rightarrow z$ as $n_k \rightarrow \infty$. Since f is accessible we can find an su -path σ connecting z to v . For n_k large enough we let γ_{n_k} be a J -legged su -path connecting $f^{n_k}(v)$ to z of length at most 1, where J is given by Proposition 31. We then let $T(\gamma_{n_k}) : \mathbb{P}E_{f^{n_k}(v)}^u \rightarrow \mathbb{P}E_z^u$ be the map obtained by composing the s - and u -holonomies along γ_{n_k} from $f^{n_k}(v)$ to z .

Let

$$A_{n_k} = T(\sigma) \circ T(\gamma_{n_k}) \circ \mathbb{P}Df_v^{n_k} : \mathbb{P}E_v^u \rightarrow \mathbb{P}E_v^u.$$

The holonomy invariance of the disintegration of the $\mathbb{P}Df$ -invariant measure μ implies that $(A_{n_k})_*\mu_v = \mu_v$ for each n_k . Choose a linear identification of $\mathbb{P}E_v^u$ with the real projective space \mathbb{RP}^{n-1} . Then A_{n_k} gives an element of the projective linear group $PSL(n, \mathbb{R})$ for each n_k . Since the transformations $T(\gamma_{n_k})$ have uniformly bounded norm together with their inverses, and since there exist unit vectors $w_1, w_2 \in E_v^u$ such that $\frac{\|Df^{n_k}(w_1)\|}{\|Df^{n_k}(w_2)\|} \rightarrow \infty$, we conclude that the sequence of transformations $\{A_{n_k}\}$ is not contained in any compact subset of $PSL(n, \mathbb{R})$. Hence, after passing to a further subsequence if necessary, there is a quasi-projective transformation Q of \mathbb{RP}^{n-1} such that A_{n_k} converges to Q on the complement of a proper linear subspace V of \mathbb{RP}^{n-1} (see [16]). Furthermore the image of Q is a proper linear subspace L of \mathbb{RP}^{n-1} .

Thus there is a proper linear subspace V of $\mathbb{P}E_v^u$ such that on the complement of V , A_{n_k} converges pointwise to a continuous map which has image contained inside of a proper subspace L of $\mathbb{P}E_v^u$. Since $(A_{n_k})_*\mu_v = \mu_v$ for every n_k , this shows that μ_v is supported on the union $V \cup L$ of two proper subspaces of $\mathbb{P}E_v^u$. Consider any point $w \in \text{supp}(\mu_v)$. By Lemma 32 there is a collection of su -loops $\gamma_1, \dots, \gamma_\ell$ based at v such that the collection of points $\{T(\gamma_i)(w)\}_{i=1}^\ell$ is δ -dense in $\mathbb{P}E_v^u$. But by the holonomy invariance of the disintegration of μ , if γ is an su -loop

based at v then $T(\gamma)(w) \in \text{supp}(\mu_v) \subset V \cup L$. This proves that the union $V \cup L$ of two proper subspaces of $\mathbb{P}E_v^\mu$ is δ -dense in $\mathbb{P}E_v^\mu$, which contradicts our choice of δ . Thus $Df|_{E^\mu}$ is uniformly quasiconformal.

REFERENCES

- [1] F. Abdenur and M. Viana. Flavors of partial hyperbolicity. Preprint <http://w3.impa.br/~viana/out/flavors.pdf>.
- [2] Artur Avila, Jimmy Santamaria, and Marcelo Viana. Holonomy invariance: rough regularity and applications to Lyapunov exponents. *Astérisque*, (358):13–74, 2013.
- [3] Artur Avila, Marcelo Viana, and Amie Wilkinson. Absolute continuity, Lyapunov exponents and rigidity I: geodesic flows. *J. Eur. Math. Soc. (JEMS)*, 17(6):1435–1462, 2015.
- [4] Yves Benoist and François Labourie. Sur les difféomorphismes d’Anosov affines à feuilletages stable et instable différentiables. *Invent. Math.*, 111(2):285–308, 1993.
- [5] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.*, 5(5):731–799, 1995.
- [6] M. Brin and H. Karcher. Frame flows on manifolds with pinched negative curvature. *Compositio Math.*, 52(3):275–297, 1984.
- [7] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Cambridge University Press, Cambridge, 2002.
- [8] Keith Burns and Mark Pollicott. Stable ergodicity and frame flows. *Geom. Dedicata*, 98:189–210, 2003.
- [9] Keith Burns and Amie Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math. (2)*, 171(1):451–489, 2010.
- [10] Clark Butler. Rigidity of equality of Lyapunov exponents for geodesic flows. 2015. Preprint arXiv:1501.05997.
- [11] Patrick B. Eberlein. *Geometry of nonpositively curved manifolds*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.
- [12] Yong Fang. Smooth rigidity of uniformly quasiconformal Anosov flows. *Ergodic Theory Dynam. Systems*, 24(6):1937–1959, 2004.
- [13] Yong Fang. On the rigidity of quasiconformal Anosov flows. *Ergodic Theory Dynam. Systems*, 27(6):1773–1802, 2007.
- [14] Yong Fang. Quasiconformal Anosov flows and quasisymmetric rigidity of Hamenstädt distances. *Discrete Contin. Dyn. Syst.*, 34(9):3471–3483, 2014.
- [15] F. W. Gehring. Rings and quasiconformal mappings in space. *Trans. Amer. Math. Soc.*, 103:353–393, 1962.
- [16] I. Ya. Gol’dsheid and G. A. Margulis. Lyapunov exponents of a product of random matrices. *Uspekhi Mat. Nauk*, 44(5(269)):13–60, 1989.
- [17] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213. Princeton Univ. Press, Princeton, N.J., 1981.
- [18] Boris Hasselblatt. Periodic bunching and invariant foliations. *Math. Res. Lett.*, 1(5):597–600, 1994.

- [19] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.
- [20] J.-L. Journé. A regularity lemma for functions of several variables. *Rev. Mat. Iberoamericana*, 4(2):187–193, 1988.
- [21] Boris Kalinin and Victoria Sadovskaya. Linear cocycles over hyperbolic systems and criteria of conformality. *J. Mod. Dyn.*, 4(3):419–441, 2010.
- [22] Boris Kalinin and Victoria Sadovskaya. Cocycles with one exponent over partially hyperbolic systems. *Geom. Dedicata*, 167:167–188, 2013.
- [23] Masahiko Kanai. Differential-geometric studies on dynamics of geodesic and frame flows. *Japan. J. Math. (N.S.)*, 19(1):1–30, 1993.
- [24] A. Katok and A. Kononenko. Cocycles’ stability for partially hyperbolic systems. *Math. Res. Lett.*, 3(2):191–210, 1996.
- [25] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, 30:499–510, 1968.
- [26] A. N. Livšic and Ja. G. Sinaĭ. Invariant measures that are compatible with smoothness for transitive C -systems. *Dokl. Akad. Nauk SSSR*, 207:1039–1041, 1972.
- [27] Charles Pugh, Michael Shub, and Amie Wilkinson. Hölder foliations. *Duke Math. J.*, 86(3):517–546, 1997.
- [28] V. A. Rohlin. On the fundamental ideas of measure theory. *Mat. Sbornik N.S.*, 25(67):107–150, 1949.
- [29] Victoria Sadovskaya. On uniformly quasiconformal Anosov systems. *Math. Res. Lett.*, 12(2-3):425–441, 2005.
- [30] Michael Shub and Amie Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.*, 139(3):495–508, 2000.
- [31] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.
- [32] Dennis Sullivan. Quasiconformal homeomorphisms in dynamics, topology, and geometry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1216–1228. Amer. Math. Soc., Providence, RI, 1987.
- [33] Pekka Tukia. On quasiconformal groups. *J. Analyse Math.*, 46:318–346, 1986.
- [34] Jussi Väisälä. *Lectures on n -dimensional quasiconformal mappings*. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, 5734 S UNIVERSITY AVE, CHICAGO, IL 60637

E-MAIL: cbutler@math.uchicago.edu

UNIV PARIS DIDEROT, SORBONNE PARIS CITÉ, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, UMR 7586, CNRS, SORBONNE UNIVERSITÉS, UPMC UNIV PARIS 06, F-75013, PARIS, FRANCE

E-MAIL: disheng.xu@imj-prg.fr